Published by Institute of Physics Publishing for SISSA/ISAS
Received: *January 22, 2003*
Accepted: *February 20, 2003*

Classical gauge instantons from open strings

Marco Billó, Marialuisa Frau and Igor Pesando
Dipartimento di Fisica Teorica, Università di Torino
Istituto Nazionale di Fisica Nucleare - sezione di Torino
via P. Giuria 1, I-10125 Torino
E-mail: billo@to.infn.it, frau@to.infn.it, ipesando@to.infn.it

Francesco Fucito
Dipartimento di Fisica, Università di Roma Tor Vergata
Istituto Nazionale di Fisica Nucleare - sezione di Roma 2
Via della Ricerca Scientifica, I-00133 Roma, Italy
E-mail: fucito@roma2.infn.it

Alberto Lerda
Dipartimento di Scienze e Tecnologie Avanzate
Università del Piemonte Orientale, I-15100 Alessandria, Italy
Istituto Nazionale di Fisica Nucleare - sezione di Torino, Italy
E-mail: lerda@to.infn.it

Antonella Liccardo
Dipartimento di Scienze Fisiche, Università di Napoli
Istituto Nazionale di Fisica Nucleare - sezione di Napoli
Complesso Universitario "Monte Sant'Angelo", via Cintia, I-80126 Napoli, Italy
E-mail: antonella.liccardo@na.infn.it

Abstract: We study the D3/D(-1) brane system and show how to compute instanton corrections to correlation functions of gauge theories in four dimensions using open string techniques. In particular we show that the disks with mixed boundary conditions that are typical of the D3/D(-1) system are the sources for the classical instanton solution. This can then be recovered from simple calculations of open string scattering amplitudes in the presence of D-instantons. Exploiting this fact we also relate this stringy description to the standard instanton calculus of field theory.

Keywords: Solitons Monopoles and Instantons, D-branes, Extended Supersymmetry.

JHEP02(2003)045

Contents

1. Introduction **1**

2. A review of the D3/D(−1) system **5**
2.1 Broken and unbroken supersymmetries 6
2.2 Massless spectrum 7

3. Effective actions and ADHM measure on moduli space **11**

4. The instanton solution from mixed disks **18**
4.1 The gauge vector profile 18
4.2 Insertions of the translational zero-modes 22

5. The superinstanton profile **23**
5.1 Unbroken supersymmetries 23
5.2 Broken supersymmetries 27

6. String amplitudes and instanton calculus **28**

A. Notations and conventions **34**

B. A short review of the ADHM construction and of zero modes around an instanton background **37**

C. Subleading order of the instanton profile in the $\alpha' \to 0$ limit **40**

1. Introduction

Recently a lot of effort has been put in investigating various properties of (supersymmetric) field theories using string theory and in particular D-branes. At the same time, a similar effort has been devoted to extend and "lift" to string theory many of the methods that have been developed over the years to study field theories. As a result of these investigations, a strong and fruitful relation between string and field theory has been established.

Quite generally one can say that in the limit of infinite tension ($\alpha' \to 0$) a string theory reduces to an effective field theory with gauge interactions unified with gravity. Even if the precise dictionary between string and field theory is not always straightforward, the simple idea of taking $\alpha' \to 0$ has been throroughly exploited to investigate the perturbative sector of various field theories using string techniques which, indeed, turned out to be very efficient computational tools (see e.g. ref. [1]). In this perturbative framework, one typically starts from string scattering amplitudes computed on a Riemann surface Σ of a given topology. In general, a N-point string amplitude $\mathcal{A}_N$ is obtained from the correlation function among N vertex operators $V_{\phi_1}, \ldots, V_{\phi_N}$, each of which describes the emission of a field ϕ_i of the string spectrum from the world-sheet. Schematically, we have

$$\mathcal{A}_N = \int_\Sigma \langle V_{\phi_1} \cdots V_{\phi_N} \rangle_\Sigma \tag{1.1}$$

where the integral is over the positions of the vertex operators and the moduli of Σ with an appropriate measure, and the symbol $\langle \cdots \rangle_\Sigma$ denotes the vacuum expectation value with respect to the (perturbative) vacuum represented by Σ.

Let us now focus on the simplest world-sheets, namely the sphere for closed strings and the disk for open strings, and let us distinguish in the vertex V_ϕ the polarization ϕ from the operator part by writing

$$V_\phi = \phi \; \mathcal{V}_\phi \,. \tag{1.2}$$

Then, for any closed string field $\phi_{\rm closed}$ we have

$$\langle \mathcal{V}_{\phi_{\rm closed}} \rangle_{\rm sphere} = 0 \,, \tag{1.3}$$

and for any open string field $\phi_{\rm open}$ we have

$$\langle \mathcal{V}_{\phi_{\rm open}} \rangle_{\rm disk} = 0 \,. \tag{1.4}$$

The relations (1.3) and (1.4) imply that the closed and open strings do not possess tadpoles on the sphere and the disk respectively; hence these are the appropriate world-sheets to describe the classical trivial vacua around which the ordinary perturbation theory is performed, but clearly they are inadequate to describe classical non-perturbative backgrounds.

However, after the discovery of D-branes [2] the perspective has drastically changed and nowadays also some non-perturbative properties can be studied in string theory. The key point is that the Dp branes are p-dimensional extended configurations of type-II and type-I string theory that, despite their non-perturbative nature, admit a perturbative description. In fact, they can be represented by closed strings in which the left and right movers are suitably identified [3]. Such an identification is equivalent to insert a boundary on the closed string world-sheet and prescribe suitable boundary reflection rules for the string coordinates [4]. Thus, the simplest world-sheet topology for closed strings in the presence of a Dp brane is that of a disk with $(p+1)$ longitudinal and $(9-p)$ transverse boundary conditions. Moreover, due to the boundary reflection rules, on such a disk we have, in general,

$$\langle \mathcal{V}_{\phi\,{\rm closed}} \rangle_{{\rm disk}_p} \neq 0 \,. \tag{1.5}$$

A Dp brane can also be represented by a boundary state $|{\rm D}p\rangle$, which is a non-perturbative state of the closed string that inserts a boundary on the world-sheet and enforces on it the appropriate identifications between left and right movers (for a review on the boundary state formalism, see for example ref. [5]). If we denote by $|\phi_{\rm closed}\rangle$ the physical state associated to the vertex operator $\mathcal{V}_{\phi_{\rm closed}}$, we can rewrite (1.5) as follows

$$\langle \phi_{\,\rm closed} | {\rm D}p \rangle \neq 0 \,. \tag{1.6}$$

Thus, the boundary state, or equivalently its corresponding disk, is a classical source for the various fields of the closed string spectrum. In particular, it is a source for the massless fields (like for instance the graviton $h_{\mu\nu}$) which acquire a non-trivial profile and therefore describe a non-trivial classical background. A precise relation between such a background and the boundary state has been established in refs. [6, 7]. There it has been shown that if one multiplies the massless tadpoles of $|\mathrm{D}p\rangle$ by free propagators and then takes the Fourier transform, one gets the leading term in the large distance expansion of the classical p-brane solutions carrying Ramond-Ramond charges which are non-perturbative configurations of type-II or type-I supergravity. For example, applying this procedure to the graviton tadpole

$$\langle \mathcal{V}_{h_{\mu\nu}} \rangle_{\mathrm{disk}_p} = \langle h_{\mu\nu} | \mathrm{D}p \rangle \,, \tag{1.7}$$

one obtains the metric of the Dp brane in the large distance approximation from which the complete supergravity solution can eventually be reconstructed. These arguments show that in order to describe closed strings in a D-brane background it is necessary to modify the boundary conditions of the string coordinates and, at the lowest order, consider disks instead of spheres.

A natural question at this point is whether this approach can be generalized to open strings, and in particular whether one can describe in this way the instantons of four dimensional gauge theory. To show that this is possible is one of the purposes of this paper. The crucial point is that the instantons of the (supersymmetric) gauge theories in four dimensions are non-perturbative configurations which admit a perturbative description within the realm of string theory. Thus, in a certain sense, they are the analogue for open strings of what the supergravity branes with Ramond-Ramond charges are for closed strings. In this analysis a key role is again played by the D-branes; this time, hovever, they are regarded from the open string point of view, namely as hypersurfaces spanned by the string end-points on which a (supersymmetric) gauge theory is defined. For definiteness, let us consider a stack of N D3 branes of type-IIB string theory which support on their world-volume a $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM) with gauge group $\mathrm{U}(N)$ (or $\mathrm{SU}(N)$ if we disregard the center of mass). Then, as shown in refs. [8, 9], in order to describe instantons of this gauge theory with topological charge k, one has to introduce k D(-1) branes (D-instantons) and thus consider a D3/D(-1) brane system. The role of D-instantons and their relation to the gauge theory instantons have been intensively studied from many different points of view in the last years (see for example refs. [10, 11, 12, 14, 13, 15, 16, 17, 18]; for recent reviews on this subject see refs. [20, 21, 22] and references therein). In the D3/D(-1) brane system, besides the ordinary perturbative gauge degrees of freedom represented by open strings stretching between two D3 branes, there are also other degrees of freedom that are associated to open strings with at least one end-point on the D-instantons. These extra degrees of freedom are non-dynamical parameters which, at the lowest level, can be interpreted as the moduli of the gauge (super)instantons in the ADHM construction [23]. Furthermore, in the limit $\alpha' \to 0$ the interactions of these parameters reproduce exactly the ADHM measure on the instanton moduli space [22].

JHEP02(2003)045

In this paper we further elaborate on this D-brane description of instantons and show that it is not only an efficient book-keeping device to account for the multiplicities and the transformation properties of the various instanton moduli, but also a powerful tool to extract from string theory a detailed information on the gauge instantons. First of all, we observe that the presence of different boundary conditions for the open strings of the D3/D(-1) system implies the existence of disks whose boundary is divided into different portions lying either on the D3 or on the D(-1) branes (see for example figure 1). These disks, which we call mixed disks, are characterized by the insertion of at least two vertex operators associated to excitations of strings that stretch between a D3 and a D(-1) brane (or viceversa), and clearly depend on the parameters (i.e. the moduli) that accompany these mixed vertex operators. Moreover, due to the change in the boundary conditions caused by the mixed operators, in general one can expect that

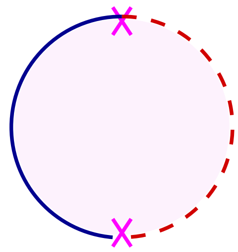

Figure 1: The simplest mixed disk with two-boundary changing operators indicated by the two crosses. The solid line represents the D3 boundary while the dashed line represents the D(-1) boundary.

$$\langle \mathcal{V}_{\phi_{\,\rm open}} \rangle_{\rm mixed\ disk} \neq 0\,. \tag{1.8}$$

In this paper we will confirm this expectation and in particular show that the massless fields of the $\mathcal{N} = 4$ gauge vector multiplet propagating on the D3 branes have non-trivial tadpoles on the mixed disks; for example, for the gauge potential A_μ, we will find that

$$\langle \mathcal{V}_{A_\mu} \rangle_{\rm mixed\ disk} \neq 0\,. \tag{1.9}$$

Furthermore, by taking the Fourier transform of these massless tadpoles after including a propagator [6, 7], we find that the corresponding space-time profile is precisely that of the classical instanton solution of the SU(N) gauge theory in the singular gauge [24, 25]. For simplicity we show this only in the case of the D3/D(-1) brane system in flat space, i.e. for instantons of the $\mathcal{N} = 4$ supersymmetry, but a similar analysis can be performed without difficulties also in orbifold backgrounds that reduce the supersymmetry to $\mathcal{N} = 2$ or $\mathcal{N} = 1$.

We can therefore assert that the mixed disks are the sources for gauge fields with an instanton profile, and thus, contrarily to the ordinary disks (see eq. (1.4)) they are the appropriate world-sheets one has to consider in order to compute instanton contributions to correlation functions within string theory. We believe that this fact helps to clarify the analysis and the prescriptions presented in refs. [11, 18] and also provides the conceptual bridge necessary to relate the D-instanton techniques of string theory with the standard instanton calculus in field theory.

This paper is organized as follows. In section 2 we review the main properties of the D3/D(-1) brane system, discuss its supersymmetries and the spectrum of its open string excitations. In section 3 we derive the effective action for the D3/D(-1) brane system by taking the field theory limit $\alpha' \to 0$ of string scattering amplitudes on (mixed) disks. In this derivation we introduce also a string representation for the auxiliary fields that linearize the supersymmetry transformation rules, and discuss how the effective action of the D3/D(−1) system reduces to the ADHM measure on the instanton moduli space by taking a suitable scaling limit. In section 4 we present one of the main result of this paper, namely that the gauge vector field emitted from a mixed disk with two boundary changing operators is exactly the leading term in the large distance expansion of the classical instanton solution in the singular gauge. We also discuss how the complete solution can be recovered by considering mixed disks with more boundary changing insertions. In section 5 we complete our analysis by considering the other components of the $\mathcal{N}=4$ vector multiplet and obtain the full superinstanton solution from mixed disks. In the last section we show how instanton contributions to correlation functions in gauge theories can be computed using string theory methods, and also clarify the relation with the standard field theory approach. Finally, in the appendices we list our conventions, give some more technical details and briefly review the ADHM costruction of the superinstanton solution.

2. A review of the D3/D(−1) system

The k instanton sector of a four-dimensional $\mathcal{N}=4$ SYM theory with gauge group SU(N) can be described by a bound state of N D3 and k D(−1) branes [8, 9]. In this section we review the main properties of this brane system, and in particular analyze its supersymmetries and the spectrum of its open string excitations.

In the D3/D(−1) system the string coordinates $X^M(\tau,\sigma)$ and $\psi^M(\tau,\sigma)$ ($M=1,\ldots,10$) obey different boundary conditions depending on the type of boundary. Specifically, on the D(−1) brane we have Dirichlet boundary conditions in all directions, while on the D3 brane the longitudinal fields X^μ and ψ^μ ($\mu=1,2,3,4$) satisfy Neumann boundary conditions, and the transverse fields X^a and ψ^a ($a=5,\ldots,10$) obey Dirichlet boundary conditions. To fully define the system, it is necessary to specify also the reflection rules of the spin fields $S^{\dot{\mathcal{A}}}$, which transform as a Weyl spinor of SO(10) (say with negative chirality). As explained for example in ref. [3], these reflection rules must be determined consistently from the boundary conditions of the $\psi^{\mathcal{M}}$'s. Introducing $z=\exp(\tau+\mathrm{i}\sigma)$ and $\bar z=\exp(\tau-\mathrm{i}\sigma)$, and denoting with a $\widetilde{\ }$ the right-moving part, it turns out that on the D(−1) boundary

$$S^{\dot{\mathcal{A}}}(z)=\varepsilon\ \widetilde{S}^{\dot{\mathcal{A}}}(\bar z)\Big|_{z=\bar z}\ , \tag{2.1}$$

while on the D3 boundary

$$S^{\dot{\mathcal{A}}}(z)=\varepsilon'\ (\Gamma^{0123}\widetilde{S})^{\dot{\mathcal{A}}}(\bar z)\Big|_{z=\bar z}\ . \tag{2.2}$$

Here, ε and ε' are signs that distinguish between branes and anti-branes. However, only the relative sign $\varepsilon\varepsilon'$ is relevant, and thus we loose no generality in setting $\varepsilon=1$ from now on.

Since the presence of the D3 branes breaks SO(10) to SO(4) × SO(6), we decompose the spin fields $S^{\dot{\mathcal{A}}}$ as follows

$$S^{\dot{\mathcal{A}}}\to\left(S_\alpha\,S_A,S^{\dot\alpha}\,S^A\right)\,, \tag{2.3}$$

where S_α ($S^{\dot\alpha}$) are SO(4) Weyl spinors of positive (negative) chirality, and S^A (S_A) are SO(6) Weyl spinors of positive (negative) chirality which transform in the fundamental (anti-fundamental) representation of SU(4) $\sim$ SO(6) (see appendix A for our conventions). Then, the D(-1) boundary conditions (2.1) become

$$S_\alpha(z)\,S_A(z) = \widetilde{S}_\alpha(\bar z)\,\widetilde{S}_A(\bar z)\Big|_{z=\bar z}\,, \qquad S^{\dot\alpha}(z)\,S^A(z) = \widetilde{S}^{\dot\alpha}(\bar z)\,\widetilde{S}^A(\bar z)\Big|_{z=\bar z}\,, \tag{2.4}$$

while the D3 boundary conditions (2.2) become

$$S_\alpha(z)\,S_A(z) = \varepsilon'\;\widetilde{S}_\alpha(\bar z)\,\widetilde{S}_A(\bar z)\Big|_{z=\bar z}\,, \qquad S^{\dot\alpha}(z)\,S^A(z) = -\varepsilon'\;\widetilde{S}^{\dot\alpha}(\bar z)\,\widetilde{S}^A(\bar z)\Big|_{z=\bar z}\,. \tag{2.5}$$

These reflection rules are essential in determining which supersymmetries are preserved or broken by the different branes.

2.1 Broken and unbroken supersymmetries

Let us recall that the charge q corresponding to a holomorphic current can be written in terms of the left and right bulk charges Q and $\widetilde{Q}$ as

$$q = Q - \widetilde{Q} = \frac{1}{2\pi\mathrm{i}}\left(\int dz\,j(z) - \int d\bar z\widetilde{j}(\bar z)\right)\,, \tag{2.6}$$

where the z ($\bar z$) integral is over a semicircle of constant radius in the upper (lower) half complex plane. The charge q is conserved at the boundary if the following condition

$$j(z) = \widetilde{j}(\bar z)\Big|_{\bar z = z} \tag{2.7}$$

holds. On the contrary, the other combination of bulk charges

$$q' = Q + \widetilde{Q} = \frac{1}{2\pi\mathrm{i}}\left(\int dz\,j(z) + \int d\bar z\widetilde{j}(\bar z)\right) \tag{2.8}$$

is broken by the boundary conditions (2.7). In this case, when the integration contours are deformed to real axis, the integrand does not vanish and thus it contributes to q' with the following amount

$$\int_{\text{boundary}} dx\,(j + \widetilde{j})\Big|_{\bar z = z \equiv x}\,. \tag{2.9}$$

This corresponds to the integrated insertion on the boundary of the massless vertex operator $(j + \widetilde{j})(x)$ which describes the Goldstone field associated to the broken symmetry generated by q'.

Let us now return to the D3/D(-1) system, and consider the bulk supercharges

$$Q^{\dot{\mathcal{A}}} = \frac{1}{2\pi\mathrm{i}}\int dz\,j^{\dot{\mathcal{A}}}(z)\,, \qquad \widetilde{Q}^{\dot{\mathcal{A}}} = \frac{1}{2\pi\mathrm{i}}\int d\bar z\;\widetilde{j}^{\dot{\mathcal{A}}}(\bar z)\,, \tag{2.10}$$

where $j^{\dot{\mathcal{A}}}$ ($\widetilde{j}^{\dot{\mathcal{A}}}$) is the left (right) supersymmetry current. In the $(-1/2)$ picture, we simply have

$$j^{\dot{\mathcal{A}}}(z) = S^{\dot{\mathcal{A}}}(z)\;\mathrm{e}^{-\frac{1}{2}\phi(z)} \tag{2.11}$$

(and similarly for the right moving current) where ϕ is the chiral boson of the superghost fermionization formulas [26].

Decomposing the spin field as in (2.3), and using the reflection rules (2.4) and (2.5), from the previous analysis it is easy to conclude that for $\varepsilon' = -1$

- the charge $Q^{\dot\alpha A} - \widetilde{Q}^{\dot\alpha A}$ is preserved both on the D3 and on the D(-1) boundary. Adopting the same notation as in [18], we denote by $\bar\xi_{\dot\alpha A}$ the fermionic parameters of the supersymmetry transformations generated by this charge;

- the charge $Q^{\dot\alpha A} + \widetilde{Q}^{\dot\alpha A}$ is broken on both types of boundaries. The corresponding parameter is denoted by $\rho_{\dot\alpha A}$;

- the charge $Q_{\alpha A} - \widetilde{Q}_{\alpha A}$ is preserved on the D(-1) boundary but is broken on the D3 boundary. The corresponding parameter is denoted by $\xi^{\alpha A}$;

- the charge $Q_{\alpha A} + \widetilde{Q}_{\alpha A}$ is preserved on the D3 boundary but is broken by the D(-1). The corresponding parameter is denoted by $\eta^{\alpha A}$.

If $\varepsilon' = 1$, the chiralities get exchanged and the charges $Q_{\alpha A} - \widetilde{Q}_{\alpha A}$ and $Q_{\alpha A} + \widetilde{Q}_{\alpha A}$ are respectively preserved and broken on both boundaries, while the charges $Q^{\dot\alpha A} - \widetilde{Q}^{\dot\alpha A}$ and $Q^{\dot\alpha A} + \widetilde{Q}^{\dot\alpha A}$ are preserved only on the D(-1) boundary and on the D3 boundary respectively. This exchange of chiralities is consistent with the fact that the two cases $\varepsilon' = \mp 1$ correspond to instanton and anti-instanton configurations in the four-dimensional gauge theory.

2.2 Massless spectrum

In the D3/D(-1) brane system there are four different kinds of open strings: those stretching between two D3-branes (3/3 strings in the following), those having both ends on a D(-1)-brane ($(-1)/(-1)$ strings), and finally those which start on a D(-1) and end on a D3 brane or vice-versa ($(-1)/3$ or $3/(-1)$ strings).

Let us first consider the 3/3 strings. In the NS sector at the massless level we find a gauge vector A^μ and six scalars φ^a which can propagate in the four longitudinal directions of the D3 brane. The corresponding vertex operators (in the (-1) superghost picture) are

$$V_A^{(-1)}(z) = A^\mu(p)\, \mathcal{V}_{A^\mu}^{(-1)}(z;p)\,, \tag{2.12}$$

$$V_\varphi^{(-1)}(z) = \varphi^a(p)\, \mathcal{V}_{\varphi^a}^{(-1)}(z;p)\,, \tag{2.13}$$

where

$$\mathcal{V}_{A^\mu}^{(-1)}(z;p) = \frac{1}{\sqrt{2}}\,\psi_\mu(z)\,\mathrm{e}^{-\phi(z)}\,\mathrm{e}^{\mathrm{i}p_\nu X^\nu(z)}\,, \tag{2.14}$$

$$\mathcal{V}_{\varphi^a}^{(-1)}(z;p) = \frac{1}{\sqrt{2}}\,\psi_a(z)\,\mathrm{e}^{-\phi(z)}\,\mathrm{e}^{\mathrm{i}p_\nu X^\nu(z)} \tag{2.15}$$

with p_ν being the longitudinal incoming momentum. Here we have taken the convention that $2\pi\alpha' = 1$; in the next section when we compute string scattering amplitudes we will reinstate the appropriate dimensional factors.

In the R sector at the massless level we find two gauginos, $\Lambda^{\alpha A}$ and $\bar\Lambda_{\dot\alpha A}$, that have opposite SO(4) chirality and transform respectively in the fundamental and anti-fundamental representation of SU(4). In the $(-1/2)$ picture, the gaugino vertex operators are

$$V_{\Lambda}^{(-1/2)}(z) = \Lambda^{\alpha A}(p)\,\mathcal{V}_{\Lambda^{\alpha A}}^{(-1/2)}(z;p)\,, \tag{2.16}$$
$$V_{\bar\Lambda}^{(-1/2)}(z) = \bar\Lambda_{\dot\alpha A}(p)\,\mathcal{V}_{\bar\Lambda_{\dot\alpha A}}^{(-1/2)}(z;p)\,, \tag{2.17}$$

where

$$\mathcal{V}_{\Lambda^{\alpha A}}^{(-1/2)}(z;p) = S_\alpha(z)\,S_A(z)\,\mathrm{e}^{-\frac{1}{2}\phi(z)}\,\mathrm{e}^{\mathrm{i}p_\nu X^\nu(z)}\,, \tag{2.18}$$
$$\mathcal{V}_{\bar\Lambda_{\dot\alpha A}}^{(-1/2)}(z;p) = S^{\dot\alpha}(z)\,S^A(z)\,\mathrm{e}^{-\frac{1}{2}\phi(z)}\,\mathrm{e}^{\mathrm{i}p_\nu X^\nu(z)}\,. \tag{2.19}$$

The massless fields introduced above form the $\mathcal{N}=4$ vector multiplet and are connected to each other by the sixteen supersymmetry transformations which are preserved on a D3 boundary and whose parameters are $\bar\xi_{\dot\alpha A}$ and $\eta^{\alpha A}$, namely

$$\begin{aligned}
\delta A^\mu &= \mathrm{i}\bar\xi_{\dot\alpha A}(\bar\sigma^\mu)^{\dot\alpha\beta}\Lambda_\beta{}^A + \mathrm{i}\eta^{\alpha A}(\sigma^\mu)_{\alpha\dot\beta}\bar\Lambda^{\dot\beta}{}_A\,,\\
\delta\Lambda^{\alpha A} &= \frac{\mathrm{i}}{2}\eta^{\beta A}(\sigma^{\mu\nu})_\beta{}^\alpha F_{\mu\nu} + \mathrm{i}\bar\xi_{\dot\beta B}(\bar\sigma^\mu)^{\dot\beta\alpha}(\Sigma^a)^{BA}\partial_\mu\varphi_a\,,\\
\delta\bar\Lambda_{\dot\alpha A} &= \frac{\mathrm{i}}{2}\bar\xi_{\dot\beta A}(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}F_{\mu\nu} - \mathrm{i}\eta^{\beta B}(\sigma^\mu)_{\beta\dot\alpha}(\bar\Sigma^a)_{BA}\partial_\mu\varphi_a\,,\\
\delta\varphi^a &= -\mathrm{i}\bar\xi_{\dot\alpha A}(\Sigma^a)^{AB}\bar\Lambda^{\dot\alpha}{}_B + \mathrm{i}\eta^{\alpha A}(\bar\Sigma^a)_{AB}\Lambda_\alpha{}^B\,,
\end{aligned} \tag{2.20}$$

where σ and $\bar\sigma$ are the Dirac matrices of SO(4), and Σ and $\bar\Sigma$ are those of SO(6). (see appendix A for our conventions).

The transformation laws (2.20) can be obtained by reducing to four dimensions the supersymmetry transformations of the $\mathcal{N}=1$ SYM theory in ten dimensions. However, they can also be obtained directly in the string formalism by using the vertex operators (2.12)-(2.13) and computing their commutators with the supersymmetry charges that are preserved on the D3 brane. For instance, taking the vertex operator (2.16) for the gaugino $\Lambda^{\alpha A}$ and the supersymmetry charge $q^{\dot\alpha A}\equiv Q^{\dot\alpha A}-\widetilde{Q}^{\dot\alpha A}$, both in the $(-1/2)$ picture, we have

$$\begin{aligned}
\Big[\bar\xi_{\dot\alpha A}q^{\dot\alpha A}, V_\Lambda^{(-1/2)}(z)\Big] &= \bar\xi_{\dot\alpha A}\oint_z\frac{dy}{2\pi\mathrm{i}}\,j^{\dot\alpha A}(y)V_\Lambda^{(-1/2)}(z)\\
&= -\bar\xi_{\dot\alpha A}\Lambda^{\beta B}\oint_z\frac{dy}{2\pi\mathrm{i}}\Big(S^{\dot\alpha}(y)S^A(y)\mathrm{e}^{-\frac{1}{2}\phi(y)}\Big)\Big(S_\beta(z)S_B(z)\mathrm{e}^{-\frac{1}{2}\phi(z)}\mathrm{e}^{\mathrm{i}p_\nu X^\nu(z)}\Big)\\
&= \Big(-\mathrm{i}\bar\xi_{\dot\alpha A}(\bar\sigma^\mu)^{\dot\alpha}{}_\beta\Lambda^{\beta A}\Big)\,\frac{1}{\sqrt{2}}\psi_\mu(z)\mathrm{e}^{-\phi(z)}\mathrm{e}^{\mathrm{i}p_\nu X^\nu(z)}
\end{aligned} \tag{2.21}$$

where in the last step we have used the contraction formulas (A.19). Comparing with (2.12), we recognize in the last line of (2.21) the vertex operator of a gauge boson with polarization

$$\delta_{\bar\xi}A^\mu = \mathrm{i}\bar\xi_{\dot\alpha A}(\bar\sigma^\mu)^{\dot\alpha\beta}\Lambda_\beta{}^A \tag{2.22}$$

in agreement with the first of eqs. (2.20). Thus, we can schematically write (2.21) as follows

$$\big[\bar\xi\,q, V_\Lambda\big] = V_{\delta_{\bar\xi}A}\,. \tag{2.23}$$

By proceeding in this way with all other vertex operators, we can reconstruct the entire transformation rules (2.20). Since in this approach the supersymmetry generators act on the vertex operators, and not on their polarizations, in order to derive the transformation rule of a given field we have to work "backwards" and apply the supercharges to the vertices of the fields which appear in the right hand side of the supersymmetry transformations.

If one considers N coincident D3-branes, all vertex operators for the 3/3 strings acquire $N \times N$ Chan-Paton factors T^I and correspondingly all polarizations will transform in the adjoint representation of $\mathrm{U}(N)$ (or $\mathrm{SU}(N)$). In this case, the supersymmetry transformation rules (2.20) must be modified accordingly, and in particular in the variation of the gauginos one must replace $F_{\mu\nu}$ with the full non-abelian field strength, the ordinary derivatives with the covariant ones and also add a term proportional to $\left[\varphi^a, \varphi^b\right]$.

Let us now consider the $(-1)/(-1)$ strings. Since now there are no longitudinal Neumann directions, the states of these strings do not carry any momentum, and thus they correspond more to moduli rather than to dynamical fields. In the NS sector we find ten bosonic moduli. Even if they are all on the same footing, for later purposes it is convenient to distinguish them into four a^μ (corresponding to the longitudinal directions of the D3 branes) and six χ^a (corresponding to the transverse directions to the D3's). Their vertex operators (in the (-1) superghost picture) read

$$V_a^{(-1)}(z) = \frac{a^\mu}{\sqrt{2}}\,\psi_\mu(z)\,\mathrm{e}^{-\phi(z)}\,, \tag{2.24}$$

$$V_\chi^{(-1)}(z) = \frac{\chi^a}{\sqrt{2}}\,\psi_a(z)\,\mathrm{e}^{-\phi(z)}\,. \tag{2.25}$$

In the R sector of the $(-1)/(-1)$ strings we find sixteen fermionic moduli which are conventionally denoted by $M^{\alpha A}$ and $\lambda_{\dot\alpha A}$, and correspond to the following vertex operators (in the $(-1/2)$ superghost picture)

$$V_M^{(-1/2)}(z) = M^{\alpha A}\,S_\alpha(z)\,S_A(z)\,\mathrm{e}^{-\frac{1}{2}\phi(z)}\,, \tag{2.26}$$

$$V_\lambda^{(-1/2)}(z) = \lambda_{\dot\alpha A}\,S^{\dot\alpha}(z)\,S^A(z)\,\mathrm{e}^{-\frac{1}{2}\phi(z)}\,. \tag{2.27}$$

The moduli we have introduced so far are related to each other by the sixteen supersymmetry transformations which are preserved on a D(-1) boundary. These can be obtained by reducing to zero dimensions the $\mathcal{N}=1$ supersymmetry transformations of the SYM theory in ten dimensions. However, since we will be ultimately interested in discussing the instanton properties of the four-dimensional gauge theory living on the D3 branes, we write only the moduli transformations which are preserved also by a D3 boundary and whose parameters have been denoted by $\bar\xi_{\dot\alpha A}$. They are

$$\begin{aligned}
\delta_{\bar\xi}\, a^\mu &= \mathrm{i}\,\bar\xi_{\dot\alpha A}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,M_\beta{}^A\,,\\
\delta_{\bar\xi}\,\chi^a &= -\,\mathrm{i}\,\bar\xi_{\dot\alpha A}\,(\Sigma^a)^{AB}\,\lambda^{\dot\alpha}{}_B\,,\\
\delta_{\bar\xi}\, M^{\alpha A} &= 0\,, \qquad \delta_{\bar\xi}\,\lambda_{\dot\alpha A} = 0\,.
\end{aligned} \tag{2.28}$$

Also these supersymmetry transformations can be obtained by commuting the charge $q^{\dot\alpha A}$ with the vertex operators of the various moduli, in complete analogy with what we have shown in (2.21). For example, we have

$$\left[\bar{\xi}\, q, V_M\right] = V_{\delta_{\bar{\xi}}\, a}\,. \tag{2.29}$$

If we consider a superposition of k D(-1) branes, the vertex operators (2.25)-(2.27) acquire $k \times k$ Chan-Paton factors t^U and the associated moduli an index in the adjoint representation of U(k). Moreover, the supersymmetry transformations of the fermionic moduli $M^{\alpha A}$ and $\lambda_{\dot{\alpha} A}$ get modified and become

$$\begin{aligned} \delta_{\bar{\xi}}\, M^{\alpha A} &= -\frac{1}{2}\bar{\xi}_{\dot{\beta} B}\, (\bar{\sigma}^{\mu})^{\dot{\beta}\alpha}\, (\Sigma^a)^{BA}\, [\chi_a, a_\mu]\;, \qquad (2.30)\\ \delta_{\bar{\xi}}\, \lambda_{\dot{\alpha} A} &= \frac{1}{4}\,\bar{\xi}_{\dot{\alpha} B}\, (\bar{\Sigma}^{ab})^{B}{}_{A}\, [\chi_a, \chi_b] + \frac{1}{4}\,\bar{\xi}_{\dot{\beta} A}\, (\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\, [a_\mu, a_\nu]\;. \qquad (2.31) \end{aligned}$$

Notice that these transformations being non linear in the moduli cannot be obtained using the vertex operator approach previously discussed. However, in the next section, we will show that this is actually possible after introducing suitable auxiliary fields.

Finally, let us consider the $3/(-1)$ and $(-1)/3$ strings which are characterized by the fact that four directions (those that are longitudinal to the D3 brane) have mixed boundary conditions. These conditions forbid any momentum and imply that in the NS sector the fields ψ^μ have integer-moded expansions with zero-modes that represent the SO(4) Clifford algebra. Therefore, the massless states of this sector are organized in two bosonic Weyl spinors of SO(4) which we denote by w and $\bar{w}$ respectively. The chirality of these spinors is fixed by the GSO projection, and depends on whether the D(-1) brane represents an instanton or an anti-instanton. In the instanton case, i.e. for $\varepsilon' = -1$ in (2.5), it turns out that w and $\bar{w}$ must be anti-chiral, and thus the corresponding vertex operators (in the (-1) superghost picture) are

$$\begin{aligned} V_w^{(-1)}(z) &= w_{\dot{\alpha}}\, \Delta(z)\, S^{\dot{\alpha}}(z)\, \mathrm{e}^{-\phi(z)}\,,\\ V_{\bar{w}}^{(-1)}(z) &= \bar{w}_{\dot{\alpha}}\, \bar{\Delta}(z)\, S^{\dot{\alpha}}(z)\, \mathrm{e}^{-\phi(z)}\,. \end{aligned} \tag{2.32}$$

Here $\Delta(z)$ and $\bar{\Delta}(z)$ are the bosonic twist and anti-twist fields with conformal dimension $1/4$, that change the boundary conditions of the X^μ coordinates from Neumann to Dirichlet and vice-versa by introducing a cut in the world-sheet [27].[1]

In the R sector of the $3/(-1)$ and $(-1)/3$ strings the fields ψ^μ have half-integer mode expansions so that there are fermionic zero-modes only in the six common transverse directions. Thus, the massless states of the R sector form two fermionic Weyl spinors of SO(6)

[1] The fact that w and $\bar{w}$ must be anti-chiral can be understood by observing that the vertices (2.32) are local with respect to the supercurrent $j^{\dot{\alpha} A}(z)$ associated to the only conserved supercharges $q^{\dot{\alpha} A}$ of the D3/D(-1) strings. Indeed, using the OPE's summarized in appendix A, we have

$$\begin{aligned} j^{\dot{\alpha} A}(z)\; V_w^{(-1)}(y) &= \left[S^{\dot{\alpha}}(z)\, S^A(z)\, \mathrm{e}^{-\frac{1}{2}\phi(z)}\right] \left[w_{\dot{\beta}}\, \Delta(y)\, S^{\dot{\beta}}(y)\, \mathrm{e}^{-\phi(y)}\right]\\ &\sim \frac{1}{(z-y)} \left[w^{\dot{\alpha}}\, \Delta(y)\, S^A(y)\, \mathrm{e}^{-\frac{3}{2}\phi(y)}\right] + \cdots \end{aligned}$$

where the ellipses stand for regular terms. If one had chosen the other chirality (corresponding to chiral moduli w_α and $\bar{w}_\alpha$), one would have obtained a branch cut in the OPE with the supercurrent $j^{\dot{\alpha} A}(z)$ and thus locality would have been spoiled. On the contrary, the chiral moduli would be local with respect to the supercurrent $j_{\alpha A}(z)$ that is conserved for an anti-instanton (i.e. for $\varepsilon' = -1$ in (2.5)).

JHEP02(2003)045

which we denote by μ and $\bar{\mu}$ respectively. Again, it is the GSO projection, together with the requirement of locality with respect to the conserved supercurrent, that fixes the SO(6) chirality of μ and $\bar{\mu}$. The appropriate choice for instanton configurations is that they must transform in the fundamental representation of SU(4) so that their vertices (in the $(-1/2)$ picture) are

$$\begin{aligned} V_{\mu}^{(-1/2)}(z) &= \mu^A \, \Delta(z) \, S_A(z) \, \mathrm{e}^{-\frac{1}{2}\phi(z)} \, , \\ V_{\bar{\mu}}^{(-1/2)}(z) &= \bar{\mu}^A \, \bar{\Delta}(z) \, S_A(z) \, \mathrm{e}^{-\frac{1}{2}\phi(z)} \, . \end{aligned} \tag{2.33}$$

In the presence of N D3 and k D(-1) branes, the vertices (2.32) and (2.33) acquire Chan-Paton factors ζ_{ui} and $\bar{\zeta}^{ui}$ transforming, respectively, in the bifundamental representations $\mathbf{N} \times \mathbf{k}$ and $\bar{\mathbf{N}} \times \bar{\mathbf{k}}$ of the gauge groups.

The unbroken supersymmetries of the D3/D(-1) system act on w and μ by the following transformations

$$\delta_{\bar{\xi}} \, w_{\dot{\alpha}} = - \, \mathrm{i} \, \bar{\xi}_{\dot{\alpha}A} \, \mu^A \, , \tag{2.34}$$

$$\delta_{\bar{\xi}} \, \mu^A = - \frac{1}{\sqrt{2}} \, \bar{\xi}_{\dot{\alpha}B} \, (\Sigma^a)^{BA} \, w^{\dot{\alpha}} \, \chi_a \, , \tag{2.35}$$

and similarly for $\bar{w}_{\dot{\alpha}}$ and $\bar{\mu}^A$. The linear supersymmetry transformation (2.34) can be obtained in the string operator formalism by commuting the charge $q^{\dot{\alpha}A}$ with the vertex operator V_μ; indeed we have

$$\left[\bar{\xi} \, q, V_\mu\right] = V_{\delta_{\bar{\xi}} w} \, . \tag{2.36}$$

On the contrary, we have

$$\left[\bar{\xi} \, q, V_w\right] = 0 \, , \tag{2.37}$$

and to derive the non-linear transformation (2.35) from the string vertex operators suitable auxiliary fields are required. Furthermore, the presence of w and $\bar{w}$ modifies the supersymmetry transformation of $\lambda_{\dot{\alpha}A}$ by a non-linear term

$$\delta_{\bar{\xi}} \, \lambda_{\dot{\alpha}A} \sim \bar{\xi}_{\dot{\alpha}A} \, \bar{w} w \, , \tag{2.38}$$

which also requires auxiliary fields in order to be derived in the string operator formalism. We conclude by mentioning that under the eight supercharges $q'_{\alpha A}$ that are preserved by the D3 branes but are broken by the D-instantons, the moduli w, $\bar{w}$, μ and $\bar{\mu}$ are invariant and that $[\eta \, q', V_w] = 0$.

3. Effective actions and ADHM measure on moduli space

In this section we compute the (tree-level) string amplitudes in the D3/D(-1) system by using the vertex operators previously introduced, and discuss the field theory limit $\alpha' \to 0$ that yields the effective actions and the ADHM measure on the instanton moduli space.

As a first example, let us consider the (color ordered) amplitude among one gauge boson and two gauginos of the 3/3 strings. This is obtained by inserting the vertex operators (2.12), (2.16) and (2.17) on a disk representing N D3 branes and is given by

$$\begin{aligned}\mathcal{A}_{(\bar\Lambda A\Lambda)} &= \left\langle\!\left\langle V_{\bar\Lambda}^{(-1/2)}\, V_A^{(-1)}\, V_\Lambda^{(-1/2)}\right\rangle\!\right\rangle \\ &\equiv C_4 \int \frac{\prod_i dz_i}{dV_{123}} \left\langle V_{\bar\Lambda}^{(-1/2)}(z_1)\, V_A^{(-1)}(z_2)\, V_\Lambda^{(-1/2)}(z_3)\right\rangle .\end{aligned} \tag{3.1}$$

In this expression dV_{abc} is the projective invariant volume element

$$dV_{abc} = \frac{dz_a\, dz_b\, dz_c}{(z_a - z_b)(z_b - z_c)(z_c - z_a)} \tag{3.2}$$

and the prefactor C_4 represents the topological normalization of a disk amplitude with the boundary conditions of a D3 brane. In general, the normalization C_{p+1} for disk amplitudes on a Dp brane can be determined using for example the unitarity methods of ref. [28], and if we take $(2\pi\alpha')^{1/2}$ as the unit of length, it reads

$$C_{p+1} = \frac{1}{2\pi^2\alpha'^2}\, \frac{1}{x_{p+1}\, g_{p+1}^2} \tag{3.3}$$

where g_{p+1} is the coupling constant of the $(p+1)$-dimensional gauge theory living on the brane world-volume which is given by

$$g_{p+1}^2 = 4\pi \left(4\pi^2\alpha'\right)^{\frac{p-3}{2}} g_s \tag{3.4}$$

in terms of the string coupling constant g_s, and x_{p+1} is the Casimir invariant of the fundamental representation of the gauge group of the Dp branes. Here we follow the standard conventions and normalize the SU(N) generators T^I on the D3 branes with $x_4 = 1/2$, i.e.

$$\mathrm{Tr}(T^I\, T^J) = \frac{1}{2}\, \delta^{IJ} \tag{3.5}$$

and the U(k) generators t^U on the D-instantons with $x_0 = 1$, i.e.[2]

$$\mathrm{tr}(t^U\, t^V) = \delta^{UV} . \tag{3.6}$$

With this choice we have

$$C_4 = \frac{1}{\pi^2\alpha'^2}\, \frac{1}{g_{\mathrm{YM}}^2} \tag{3.7}$$

where $g_{\mathrm{YM}}^2 \equiv g_4^2 = 4\pi g_s$ is the gauge coupling constant of the four-dimensional SYM theory, and

$$C_0 = \frac{1}{2\pi^2\alpha'^2}\, \frac{1}{g_0^2} = \frac{2\pi}{g_s} = \frac{8\pi^2}{g_{\mathrm{YM}}^2} . \tag{3.8}$$

Notice that the normalization C_4 of a D3 amplitude is dimensionful, whereas the normalization C_0 of a D-instanton amplitude is dimensionless and equal to the action of a gauge instanton.

[2] In this way the one-instanton case ($k = 1$) can be simply obtained by removing the trace symbol from all formulas without extra numerical factors.

To compute the amplitude (3.1), we must further remember that in section 2 all vertex operators have been written with the convention that $2\pi\alpha' = 1$, and thus suitable dimensional factors must be reinstated in the calculation. This can be systematically done by rescaling all bosonic fields of the NS sector by a factor of $(2\pi\alpha')^{1/2}$ so that they acquire the canonical dimension of $(\text{length})^{-1}$, and by rescaling all fermionic fields of the R sector by a factor of $(2\pi\alpha')^{3/4}$ so that they acquire the canonical dimension of $(\text{length})^{-3/2}$. Taking all these normalization factors into account and using the contraction formulas of appendix A, we find

$$\mathcal{A}_{(\bar{\Lambda}A\Lambda)} = -\frac{2\,\mathrm{i}}{g_{\mathrm{YM}}^2}\,\mathrm{Tr}\left(\bar{\Lambda}_{\dot{\alpha}A}\,\bar{\not{A}}^{\dot{\alpha}\beta}\,\Lambda_{\beta}{}^{A}\right) \tag{3.9}$$

where the δ-function of momentum conservation is understood. The complete result is obtained by adding to (3.9) all other inequivalent color orderings, and thus the total coupling among two gauginos and one gauge boson is given by

$$-\frac{2\,\mathrm{i}}{g_{\mathrm{YM}}^2}\,\mathrm{Tr}\left(\bar{\Lambda}_{\dot{\alpha}A}\left[\bar{\not{A}}^{\dot{\alpha}\beta}\,,\,\Lambda_{\beta}{}^{A}\right]\right)\,. \tag{3.10}$$

All other interactions among the massless 3/3 string modes can be computed in a similar way. After taking the limit $\alpha' \to 0$ with g_{YM} held fixed in all string amplitudes and taking their Fourier transform, one finds that their 1PI parts are encoded in the (euclidean) action of the $\mathcal{N} = 4$ SYM theory[3]

$$\begin{aligned}\mathcal{S}_{\mathrm{SYM}} = \frac{1}{g_{\mathrm{YM}}^2}\int d^4x\,\mathrm{Tr}\Big\{&\frac{1}{2}F_{\mu\nu}^{\,2} - 2\bar{\Lambda}_{\dot{\alpha}A}\,\bar{\not{\mathcal{D}}}^{\dot{\alpha}\beta}\,\Lambda_{\beta}{}^{A} + (\mathcal{D}_\mu\varphi_a)^2 - \frac{1}{2}\left[\varphi_a,\varphi_b\right]^2 - \\ &- \mathrm{i}\,(\Sigma^a)^{AB}\,\bar{\Lambda}_{\dot{\alpha}A}\big[\varphi_a,\bar{\Lambda}^{\dot{\alpha}}{}_{B}\big] - \mathrm{i}\,(\bar{\Sigma}^a)_{AB}\,\Lambda^{\alpha A}\big[\varphi_a,\Lambda_{\alpha}{}^{B}\big]\Big\}\,,\end{aligned} \tag{3.11}$$

which is invariant under the non-abelian version of the supersymmetry transformation rules (2.20).

Let us now turn to the interactions among the $(-1)/(-1)$ strings which are obtained by evaluating correlation functions on disks representing k D(-1) branes. For example, the color ordered coupling among $\lambda_{\dot{\alpha}A}$, a_μ and $M^{\alpha A}$ corresponds to

$$\mathcal{A}_{(\lambda a M)} = \left\langle\!\!\left\langle V_{\lambda}^{(-1/2)}\,V_{a}^{(-1)}\,V_{M}^{(-1/2)}\right\rangle\!\!\right\rangle \tag{3.12}$$

where the vertex operators are given in (2.24), (2.26) and (2.27) with suitable factors of $2\pi\alpha'$ inserted as discussed above in order to assign the canonical dimensions to the various fields. In (3.12) the expectation value is computed in analogy with (3.1) but now the overall normalization is C_0 given in (3.8), as is appropriate for a disk with a D(-1) boundary. After adding all color orderings, one finds that the total coupling under consideration is

$$-\frac{\mathrm{i}}{g_0^2}\,\mathrm{tr}\left(\lambda_{\dot{\alpha}A}\left[\bar{\not{a}}^{\dot{\alpha}\beta}\,,\,M_{\beta}{}^{A}\right]\right) \tag{3.13}$$

[3] Remember that in euclidean space the 1PI part of a scattering amplitude is equal to *minus* the corresponding interaction term in the action. Moreover, the terms of higher order in α' in the scattering amplitudes represent string corrections to the standard field theory.

JHEP02(2003)045

where the trace is now taken on the indices labeling the k D(-1) branes. Interestingly, the various normalization coefficients have conspired to reproduce the (dimensionful) coupling constant g_0 with no other factors of α' left over. If we proceed in a similar way and take the field theory limit $\alpha' \to 0$ with g_0 held fixed, we find that all irreducible couplings of the $(-1)/(-1)$ strings are encoded in the effective action

$$\mathcal{S}_{(-1)} = \mathcal{S}_{\rm cubic} + \mathcal{S}_{\rm quartic} \tag{3.14}$$

where

$$\mathcal{S}_{\rm cubic} = \frac{\rm i}{g_0^2}\,{\rm tr}\Big\{\lambda_{\dot\alpha A}\Big[\not{a}^{\dot\alpha\beta}, M_\beta{}^A\Big] - \frac{1}{2}\,(\Sigma^a)^{AB}\,\lambda_{\dot\alpha A}\,\big[\chi_a, \lambda^{\dot\alpha}{}_B\big] - \frac{1}{2}\,(\bar\Sigma^a)_{AB}\,M^{\alpha A}\big[\chi_a, M_\alpha{}^B\big]\Big\} \tag{3.15}$$

and

$$\mathcal{S}_{\rm quartic} = -\frac{1}{g_0^2}\,{\rm tr}\Big\{\frac{1}{4}\,[a_\mu, a_\nu]^2 + \frac{1}{2}\,[a_\mu, \chi_a]^2 + \frac{1}{4}\,[\chi_a, \chi_b]^2\Big\}\,. \tag{3.16}$$

This action, which is the reduction to zero dimensions of the $\mathcal{N} = 1$ SYM action in ten dimensions, vanishes in the abelian case of a single D(-1) brane, i.e. for $k = 1$. It is interesting to observe that the quartic interactions in (3.16) can be decoupled by means of auxiliary fields. In fact, $\mathcal{S}_{\rm quartic}$ is equivalent to

$$\begin{aligned}\mathcal{S}\,' = \frac{1}{g_0^2}\,{\rm tr}\Big\{&\frac{1}{2}\,D_c^2 + \frac{1}{2}\,D_c\,\bar\eta^c_{\mu\nu}\,[a^\mu, a^\nu] + \frac{1}{2}\,Y_{\mu a}^2 + Y_{\mu a}\,[a^\mu, \chi^a] + \\ &+ \frac{1}{4}\,Z_{ab}^2 + \frac{1}{2}\,Z_{ab}\,\big[\chi^a, \chi^b\big]\Big\}\end{aligned} \tag{3.17}$$

where $\bar\eta$ is the anti-self dual 't Hooft symbol and D, Y and Z are auxiliary fields with dimensions of $(length)^{-2}$ which reproduce the quartic couplings of (3.16) after they are eliminated through their equations of motion. It is worth remarking that, in order to decouple the interaction ${\rm tr}\,[a_\mu, a_\nu]^2$, because of Jacobi identity it is enough to introduce three independent degrees of freedom which correspond to an antisymmetric tensor $D_{\mu\nu}$ of a given duality. For definiteness we have chosen this tensor to be anti-self dual and thus have written $D_{\mu\nu} = D_c\,\bar\eta^c_{\mu\nu}$.

The cubic couplings of $\mathcal{S}\,'$ can be obtained in the string operator formalism by introducing the following vertices for the auxiliary fields (in units of $2\pi\alpha' = 1$)

$$\begin{aligned}V_D^{(0)}(z) &= \frac{1}{2}\,D_c\,\bar\eta^c_{\mu\nu}\,\psi^\nu(z)\psi^\mu(z)\,,\\ V_Y^{(0)}(z) &= Y_{\mu a}\,\psi^a(z)\psi^\mu(z)\,,\\ V_Z^{(0)}(z) &= \frac{1}{2}\,Z_{ab}\,\psi^b(z)\psi^a(z)\,.\end{aligned} \tag{3.18}$$

These NS vertices are written in the 0-superghost picture and, even if they are not BRST invariant,[4] they provide the correct structures and interactions. Fox example, the (color-

[4]The lack of BRST invariance of the vertices (3.18) should not be regarded as a serious problem since, when dealing with auxiliary fields, one is effectively working off-shell. Vertices similar to those of (3.18) (but in the (-2) superghost picture) have been considered in ref. [29].

ordered) coupling among the auxiliary field D and two a's is reproduced by

$$\mathcal{A}_{(Daa)} = \frac{1}{2}\left\langle\!\!\left\langle V_D^{(0)}\, V_a^{(-1)}\, V_a^{(-1)} \right\rangle\!\!\right\rangle = -\,\frac{1}{2g_0^2}\,\mathrm{tr}\left(D_c\,\bar{\eta}^c_{\mu\nu}\,a^\mu\,a^\nu\right) \tag{3.19}$$

where a symmetry factor of $1/2$ has been inserted to account for the presence of two alike vertices, and the auxiliary field has been rescaled with $(2\pi\alpha')$ to make it of canonical dimension. All other cubic interactions of the action (3.17) can be obtained in a similar way.

The vertex operators (3.18) are useful also because they linearize the supersymmetry transformation rules of the various moduli which can therefore be obtained completely within the string operator formalism. In fact, using the method described in section 2, one can show for example that

$$\left[\bar{\xi}\,q, V_D\right] = V_{\delta_{\bar{\xi}}\,\lambda} \tag{3.20}$$

where $V_{\delta_{\bar{\xi}}\,\lambda}$ is the vertex (2.27) with polarization

$$\delta_{\bar{\xi}}\,\lambda_{\dot{\alpha}A} = -\,\frac{1}{4}\,\bar{\xi}_{\dot{\beta}A}\,(\bar{\sigma}^{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\,D_c\,\bar{\eta}^c_{\mu\nu}\,. \tag{3.21}$$

If the auxiliary fields D_c are eliminated through their equations of motion following from $\mathcal{S}\,'$, then (3.21) reproduces exactly the last non-linear term in the supersymmetry transformation rule (2.31). Similarly, the other terms in (2.31) and (2.30) can be obtained by computing $\left[\bar{\xi}\,q, V_Z\right]$ and $\left[\bar{\xi}\,q, V_Y\right]$.

Let us now analyze the interactions of the $(-1)/3$ and $3/(-1)$ strings. In this case the novelty is represented by the fact that the vertex operators (2.32) and (2.33) contain the twist and anti-twist fields, Δ and $\bar{\Delta}$, which change the boundary conditions of the longitudinal coordinates X^μ. Thus, for consistency in any correlation function a vertex operator of the $(-1)/3$ sector must always be accompanied by one of the $3/(-1)$ sector. This gives rise to mixed disks whose boundary is divided into an even number of portions with different boundary conditions.[5] The simplest case is the mixed disk represented in figure 1 where a pair of twist/anti-twist operators divides its boundary in two portions with D3 and D(-1) boundary conditions respectively. The topological normalization for the expectation value on such a mixed disk is C_0 given in (3.8), i.e. the normalization of the lowest brane.

Let us now consider a 3-point amplitude originating from the insertion of a $(-1)/(-1)$ state on a mixed disk, like for example

$$\mathcal{A}_{(w\lambda\bar{\mu})} = \left\langle\!\!\left\langle V_w^{(-1)}\, V_\lambda^{(-1/2)}\, V_{\bar{\mu}}^{(-1/2)} \right\rangle\!\!\right\rangle . \tag{3.22}$$

This correlation function can be computed in a straightforward manner by using the OPE's of appendix A, and the result is

$$\mathcal{A}_{(w\lambda\bar{\mu})} = \frac{2\,\mathrm{i}}{g_0^2}\,\mathrm{tr}\left(w_{\dot{\alpha}}{}^u\,\lambda^{\dot{\alpha}}{}_A\,\bar{\mu}^A{}_u\right) \tag{3.23}$$

[5]String amplitudes on mixed disks have been previously analyzed in ref. [30, 31] to study the gauge interactions of the non-BPS D-particles of the type-IIB theory.

where we have explicitly indicated also the index u of the fundamental representation of SU(N) carried by the "twisted" moduli. Again all normalizations have conspired to reconstruct the coupling constant g_0 with no other factors of α' left over. Thus, this amplitude survives in the limit $\alpha' \to 0$ with g_0 fixed, and must be added to the zero-dimensional effective action $\mathcal{S}_{(-1)}$. Other terms of this effective action could arise from amplitudes involving the vertex operators (3.18) of the auxiliary fields. For example, we have

$$\begin{aligned} \mathcal{A}_{(wD\bar{w})} &= \left\langle\!\!\left\langle V_w^{(-1)}\, V_D^{(0)}\, V_{\bar{w}}^{(-1)} \right\rangle\!\!\right\rangle \\ &= \frac{1}{2g_0^2}\, \bar{\eta}^c_{\mu\nu}\, \mathrm{tr}\left(w_{\dot\alpha}^{\ \ u}\, D_c\, \bar{w}^{\dot\beta}_{\ \ u} \right) (\bar{\sigma}^{\mu\nu})^{\dot\alpha}_{\ \ \dot\beta} = \frac{2\,\mathrm{i}}{g_0^2}\, \mathrm{tr}\left(D_c\, W^c \right) \end{aligned} \tag{3.24}$$

where in the last step we have introduced the $k \times k$ matrices

$$\left(W^c\right)_j^{\ i} = w_{\dot\alpha}^{\ \ ui}\, (\tau^c)^{\dot\alpha}_{\ \ \dot\beta}\, \bar{w}^{\dot\beta}_{\ \ uj} \tag{3.25}$$

with τ^c being the Pauli matrices. We remark in passing that the coupling (3.24) modifies the field equations of D_c by a term proportional to W_c. Thus, when the auxiliary fields are eliminated from the supersymmetry transformation rule (3.21), the structure (2.38) can be reproduced.

If we proceed systematically and compute all amplitudes on mixed disks which survive in the field theory limit, we can reconstruct the following effective action for w, $\bar{w}$, μ and $\bar{\mu}$

$$\mathcal{S}\,'' = \frac{2\,\mathrm{i}}{g_0^2}\, \mathrm{tr}\left\{ \left(\bar{\mu}^A_{\ \ u} w_{\dot\alpha}^{\ \ u} + \bar{w}_{\dot\alpha u} \mu^{Au} \right) \lambda^{\dot\alpha}_A - D_c\, W^c + \frac{1}{2} (\bar{\Sigma}^a)_{AB}\, \bar{\mu}^A_{\ \ u}\, \mu^{Bu} \chi_a - \mathrm{i}\, \chi_a\, \bar{w}_{\dot\alpha u} w^{\dot\alpha u} \chi^a \right\}. \tag{3.26}$$

Notice that the auxiliary fields Y and Z do not appear in this action. In fact, all mixed amplitudes involving them vanish either at the string level, or in the field theory limit. We point out that in analogy with what we have done before, also the quartic interaction of (3.26) can be decoupled by introducing a pair of auxiliary fields $X_{\dot\alpha a}$ and $\bar{X}_{\dot\alpha a}$. Their corresponding vertex operators, which are proportional to $S^{\dot\alpha}\psi^a\Delta$ and $S^{\dot\alpha}\psi^a\bar{\Delta}$ respectively, can be used to derive the non-linear supersymmetry transformations rules (2.35) in the string operator formalism. However, since these auxiliary fields do not play any other role, we will not introduce them in our analysis.

We can summarize our findings by saying that the total effective action for the moduli produced by the D-instantons is given by

$$\mathcal{S}_{\mathrm{moduli}} = \mathcal{S}_{\mathrm{cubic}} + \mathcal{S}\,' + \mathcal{S}\,''\,. \tag{3.27}$$

As we have thoroughly discussed, the zero-dimensional action (3.27) arises from string scattering amplitudes on D(-1) branes in the limit $\alpha' \to 0$ with g_0 fixed, whereas the four-dimensional SYM action (3.11) is obtained from string amplitudes on D3 branes in the limit $\alpha' \to 0$ with g_{YM} fixed. However, as is clear from (3.4), g_{YM} and g_0 cannot be kept fixed at the same time: indeed, when $\alpha' \to 0$ either $g_{\mathrm{YM}} \to 0$ if g_0 is fixed, or $g_0 \to \infty$ if g_{YM} is fixed. This simple fact shows that while a system made of D3 and D(-1) branes

JHEP02(2003)045

is perfectly well-defined and stable at the string level, its field theory limit, instead, is more subtle and requires some care. Since we are interested in analyzing the four-dimensional SYM theory, we clearly must keep fixed $g_{\rm YM}$ when $\alpha' \to 0$, and hence we should consider the zero-dimensional moduli action in the strong coupling limit $g_0 \to \infty$. If we take this limit in a naive way, we obtain a rather trivial result because the action (3.27), which is inversely proportional to g_0^2, becomes negligible and all effects of the D-instantons inside the D3 branes disappear. However, there is another possibility that yields more interesting results: it consists in taking g_0 and (some of) the moduli to infinity. In particular, if we take

$$
\begin{array}{llll}
a = \sqrt{2}\, g_0\, a' \,, & \chi = \chi' \,, & M = \dfrac{g_0}{\sqrt{2}}\, M' \,, & \lambda = \lambda' \,, \\
D = D' \ , & Y = \sqrt{2}\, g_0\, Y' \,, & Z = g_0\, Z' \,, & \\
w = \dfrac{g_0}{\sqrt{2}}\, w' \,, & \bar{w} = \dfrac{g_0}{\sqrt{2}}\, \bar{w}' \,, & \mu = \dfrac{g_0}{\sqrt{2}}\, \mu' \,, & \bar{\mu} = \dfrac{g_0}{\sqrt{2}}\, \bar{\mu}' \,,
\end{array}
\tag{3.28}
$$

and keep the primed variables fixed when $g_0 \to \infty$, we can easily see that the moduli action (3.27) survives in the field theory limit, and becomes

$$
\begin{aligned}
S_{\rm moduli} = {\rm tr} \Big\{ & {Y'}^{2}_{\mu a} + 2\, {Y'}_{\mu a} \left[a'^{\mu}, \chi'^{a}\right] + \frac{1}{4}\, {Z'}^{2}_{ab} + {\chi'}_{a}\, {\bar{w}'}_{\dot{\alpha} u}\, w'^{\dot{\alpha} u}\, \chi'^{a} + \\
& + \frac{\rm i}{2}\, (\bar{\Sigma}^{a})_{AB}\, {\bar{\mu}'}^{A}_{\ u}\, \mu'^{Bu}\, {\chi'}_{a} - \frac{\rm i}{4}\, (\bar{\Sigma}^{a})_{AB}\, M'^{\alpha A} \left[{\chi'}_{a}, {M'}_{\alpha}^{\ B}\right] + \\
& + {\rm i} \left({\bar{\mu}'}^{A}_{\ u}\, {w'}_{\dot{\alpha}}^{\ u} + {\bar{w}'}_{\dot{\alpha} u}\, \mu'^{Au} + \left[M'^{\beta A}, a'_{\beta\dot{\alpha}}\right] \right) {\lambda'}^{\dot{\alpha}}_{\ A} - \\
& - {\rm i}\, D'_{c} \left(W'^{c} + {\rm i}\, \bar{\eta}^{c}_{\mu\nu} \left[a'^{\mu}, a'^{\nu}\right] \right) \Big\} \,.
\end{aligned}
\tag{3.29}
$$

If we integrate out the auxiliary fields Y' and Z', the action (3.29) reduces exactly to the sum of the actions S_K and S_D defined in eqs. (10.70b) and (10.70c) of ref. [22] (up to a redefinition of $\chi'_a \to -{\rm i}\, \chi'_a$). The action (3.29) provides the ADHM measure on the moduli space of the k-instanton sector of the $\mathcal{N} = 4$ SU(N) SYM theory; in particular, the equations of motion for D'_c are precisely the three non-linear ADHM constraints

$$
W'^{c} + {\rm i}\, \bar{\eta}^{c}_{\mu\nu} \left[a'^{\mu}, a'^{\nu}\right] = 0 \,,
\tag{3.30}
$$

while the equations of motion for ${\lambda'}^{\dot{\alpha}}_{A}$ are the fermionic constraints

$$
{\bar{\mu}'}^{A}_{\ u}\, {w'}_{\dot{\alpha}}^{\ u} + {\bar{w}'}_{\dot{\alpha} u}\, \mu'^{Au} + \left[M'^{\beta A}, a'_{\beta\dot{\alpha}}\right] = 0
\tag{3.31}
$$

of the ADHM construction. From now on, to avoid clutter we drop the $'$ from all moduli, but we keep the traditional notation for a' and M'.[6]

[6]The procedure to obtain the ADHM measure that we have explained consists of two distinct steps: the first is the field theory limit on the D(-1) branes, the second is the strong coupling limit accompanied by a rescaling of the D(-1) fields which survive the first step. However, it is also possible to obtain the ADHM measure directly in a single step. This can be done by using always adimensional polarizations rescaled as

In this section we have explicitly reviewed that the D3/D(-1) system accomodates all instanton moduli of a four-dimensional supersymmetric gauge theory. It is worth pointing out, however, that the ADHM measure on moduli space does not follow automatically from this construction. In fact, as we have shown, this measure emerges only by taking the field theory limit of the D3/D(-1) system in a very specific way, which includes a rescaling of some of the string moduli with the dimensionful coupling g_0, as indicated in (3.28), and the strong coupling limit $g_0 \to \infty$.

4. The instanton solution from mixed disks

The disk diagrams considered in the previous section do not exhaust all possibilities, since there exhist also mixed disks with the emission of 3/3 strings. In this and the following sections we explicitly analyze such mixed diagrams and show that they are directly related to the classical instanton solutions of the four-dimensional SYM theory. In particular we show that the D(-1) branes effectively act as a source for the various fields of the gauge supermultiplet and that the $(-1)/(-1)$ strings together with the boundary changing operators associated to the $3/(-1)$ and $(-1)/3$ strings provide the correct dependence of the instanton profile on the ADHM moduli. For simplicity we will discuss in detail only the case of instanton number $k = 1$ in a SU(N) gauge theory. However, no substantial changes occur in our analysis if one considers higher values of k. Moreover, in the following we will set again $2\pi\alpha' = 1$ since all dimensional factors cancel out in the final results.

4.1 The gauge vector profile

Let us begin by considering the emission of the gauge vector field A^I_μ from a mixed disk. The simplest diagram which can contribute to this process contains two bosonic boundary changing operators ($V_{\bar{w}}$ and V_w) and no D(-1)/D(-1) moduli, as shown in figure 2.

The amplitude (in momentum space) associated to this diagram is

$$A^I_\mu(p;\bar{w},w) = \left\langle\!\!\left\langle V^{(-1)}_{\bar{w}}\, \mathcal{V}^{(0)}_{A^I_\mu}(-p)\, V^{(-1)}_{w} \right\rangle\!\!\right\rangle \qquad (4.1)$$

where, like for any mixed disk, the expectation value is normalized with C_0. Since we want to describe the source for the emission of a gauge boson, in the correlation function (4.1) we have inserted a gluon vertex

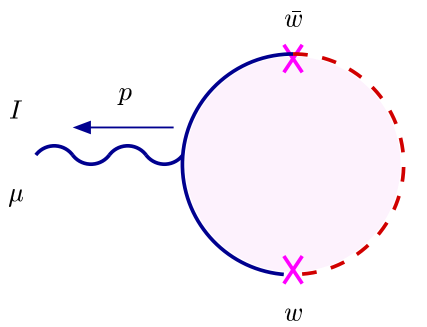

Figure 2: The mixed disk that describes the emission of a gauge vector field A^I_μ with momentum p represented by the outgoing wavy line.

follows

$$a = \left(\frac{2g_s}{\pi}\right)^{1/2} s^{\alpha}\, a' , \qquad \chi = s^{-\alpha}\,\chi' , \qquad M = \left(\frac{g_s}{2\pi}\right)^{1/2} s^{\alpha/2}\, M' , \qquad \lambda = s^{-3\alpha/2}\,\lambda' ,$$
$$D = s^{-2\alpha}\, D' , \qquad Y = \sqrt{2}\, Y' , \qquad Z = Z' ,$$
$$w = \left(\frac{g_s}{2\pi}\right)^{1/2} s^{\alpha}\, w' , \qquad \bar{w} = \left(\frac{g_s}{2\pi}\right)^{1/2} s^{\alpha}\, \bar{w}' , \qquad \mu = \left(\frac{g_s}{2\pi}\right)^{1/2} s^{\alpha/2}\, \mu' , \qquad \bar{\mu} = \left(\frac{g_s}{2\pi}\right)^{1/2} s^{\alpha/2}\, \bar{\mu}' ,$$

with $\alpha < 0$, and then letting $s \to 0$. It turns out that the action which survives in this limit is precisely given by eq. (3.29). The standard dimensions of the ADHM moduli can then be recovered by introducing suitable factors of $(2\pi\alpha')$.

operator with *outgoing* momentum and *without* polarization, so that the amplitude (4.1) carries the Lorentz structure and the quantum numbers that are appropriate for an emitted gauge vector field. Moreover, the gluon vertex is in the 0 superghost picture. This can be obtained by performing a picture changing on the vertex (2.14) and reads

$$\mathcal{V}^{(0)}_{A^I_\mu}(z;-p) = 2\mathrm{i}\, T^I\, (\partial X_\mu - \mathrm{i}\, p\cdot\psi\,\psi_\mu)\, \mathrm{e}^{-\mathrm{i}p\cdot X(z)} \tag{4.2}$$

where T^I is the adjoint $\mathrm{SU}(N)$ Chan-Paton factor.[7] The vertices for the w and $\bar{w}$ moduli are instead in the (-1) superghost picture, and are given in (2.32). However, due to the rescalings (3.28), an overall factor of $(g_s/2\pi)^{1/2}$ must be incorporated in each of these vertices in order to interprete their polarizations as the w and $\bar{w}$ moduli of the ADHM construction. Using the contraction formulas of appendix A, and taking into account (see eq. (A.21)) that

$$\left\langle \bar{\Delta}(z_1)\, \mathrm{e}^{-\mathrm{i}p\cdot X(z_2)}\, \Delta(z_3)\right\rangle = -\,\mathrm{e}^{-\mathrm{i}p\cdot x_0}\, (z_1 - z_3)^{-1/2} \tag{4.3}$$

where x_0 denotes the location of the D-instanton inside the world-volume of the D3 branes (see also eq. (A.21)), one easily finds that the amplitude (4.1) is given by

$$A^I_\mu(p;\bar{w},w) = \mathrm{i}\,(T^I)^v{}_u\, p^\nu\, \bar{\eta}^c_{\nu\mu} \left(w_{\dot{\alpha}}{}^u\, (\tau_c)^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{w}^{\dot{\beta}}{}_v\right)\, \mathrm{e}^{-\mathrm{i}p\cdot x_0} \equiv \mathrm{i}\, p^\nu\, J^I_{\nu\mu}(\bar{w},w)\, \mathrm{e}^{-\mathrm{i}p\cdot x_0} \tag{4.4}$$

where, in the last step, we have introduced the convenient notation $J^I_{\nu\mu}(\bar{w},w)$ for the moduli dependence. Note that the various factors of g_s and π's coming from the rescalings and from the normalization C_0 of the mixed disk have canceled out completely in this calculation.

As we have discussed before, the mixed disk of figure 2 represents the source in momentum space for the emission of the gauge vector field in a non-trivial background. To obtain the space-time profile of this background, we simply have to take the Fourier transform of the amplitude $A^I_\mu(p;\bar{w},w)$ after attaching to it the gluon propagator $\delta_{\mu\nu}/p^2$. Thus, the classical field associated to the mixed disk of figure 2 is

$$\begin{aligned} A^I_\mu(x) &= \int \frac{d^4p}{(2\pi)^2}\, A^I_\mu(p;\bar{w},w)\, \frac{1}{p^2}\, \mathrm{e}^{\mathrm{i}p\cdot x} \\ &= -\,2\,(T^I)^v{}_u \left(w_{\dot{\alpha}}{}^u\, (\tau_c)^{\dot{\alpha}}{}_{\dot{\beta}}\, \bar{w}^{\dot{\beta}}{}_v\right)\, \bar{\eta}^c_{\nu\mu}\, \frac{(x-x_0)^\nu}{(x-x_0)^4}\,. \end{aligned} \tag{4.5}$$

This result can also be rewritten in terms of the antisymmetric "source" tensor $J^I_{\nu\mu}$ as follows

$$A^I_\mu(x) = J^I_{\nu\mu}(\bar{w},w) \int \frac{d^4p}{(2\pi)^2}\, \frac{\mathrm{i}p^\nu}{p^2}\, \mathrm{e}^{\mathrm{i}p\cdot(x-x_0)} = J^I_{\nu\mu}(\bar{w},w)\, \partial^\nu G(x-x_0) \tag{4.6}$$

where

$$G(x-x_0) = \int \frac{d^4p}{(2\pi)^2}\, \frac{\mathrm{e}^{\mathrm{i}p\cdot(x-x_0)}}{p^2} = \frac{1}{(x-x_0)^2} \tag{4.7}$$

is the scalar massless propagator in configuration space.

The gauge field $A^I_\mu(x)$ in (4.5) depends on the $4N$ moduli $w_{\dot{\alpha}}{}^u$ and $\bar{w}_{\dot{\alpha}u}$, up to an overall phase redefinition $w \sim \mathrm{e}^{\mathrm{i}\theta} w$ and $\bar{w} \sim \mathrm{e}^{-\mathrm{i}\theta}\bar{w}$, and on the position x_0^μ of the D-instanton inside

[7]The overall factor of 2i, which is not determined by the picture changing, is fixed by requiring the appropriate normalization of the three gluon amplitude.

JHEP02(2003)045

the world-volume of the D3 branes. This amounts to $4N+3$ real parameters which are precisely those of the *unconstrained* instanton moduli space in the ADHM construction. In fact, upon enforcing the three bosonic ADHM constraints $W^c = 0$ (see eq. (3.30) for $k=1$), these parameters reduce exactly to the $4N$ moduli of the SU(N) instanton, namely the position of its center x_0^μ, its size ρ and the $4N-5$ varibles that parametrize the coset space SU(N)/S[U($N-2$) $\times$ U(1)] and specify the orientation of a SU(2) subgroup inside SU(N). To see this explicitly, let us define

$$2\rho^2 \equiv \bar{w}^{\dot\alpha}{}_u \, w_{\dot\alpha}{}^u \,, \tag{4.8}$$

and consider the three $N \times N$ matrices

$$(t_c)^u{}_v \equiv \frac{1}{2\rho^2} \left(w_{\dot\alpha}{}^u \, (\tau_c)^{\dot\alpha}{}_{\dot\beta} \, \bar{w}^{\dot\beta}{}_v \right) . \tag{4.9}$$

Then, it is not difficult to show that these matrices generate a SU(2) subalgebra of SU(N), i.e. $[t_c, t_d] = \mathrm{i}\epsilon_{cde}\, t_e$, provided the ADHM constraints $W^c = 0$ are satisfied. In conclusion, we can rewrite the gauge field (4.5) as follows

$$A^I_\mu(x) = 4\rho^2 \, \mathrm{Tr}(T^I \, t_c) \, \bar\eta^c_{\mu\nu} \, \frac{(x-x_0)^\nu}{(x-x_0)^4} \,. \tag{4.10}$$

In the case of SU(2) the indices I and c can be identified and, taking into account the trace normalization, we obtain

$$A^c_\mu(x) = 2\rho^2 \, \bar\eta^c_{\mu\nu} \, \frac{(x-x_0)^\nu}{(x-x_0)^4} \,. \tag{4.11}$$

In this expression we recognize precisely the leading term in the large distance expansion (i.e. $|x - x_0| >> \rho$) of the classical BPST SU(2) instanton [24, 25] with center x_0 and size ρ, in the so-called *singular gauge*, namely

$$\begin{aligned} A^c_\mu(x) &= 2\rho^2 \, \bar\eta^c_{\mu\nu} \, \frac{(x-x_0)^\nu}{(x-x_0)^2 \, [(x-x_0)^2 + \rho^2]} \\ &\simeq 2\rho^2 \, \bar\eta^c_{\mu\nu} \, \frac{(x-x_0)^\nu}{(x-x_0)^4} \left(1 - \frac{\rho^2}{(x-x_0)^2} + \cdots \right) . \end{aligned} \tag{4.12}$$

Notice that such a configuration has a self-dual field strength, despite the appearance of the anti self-dual 't Hooft symbols $\bar\eta^c_{\mu\nu}$.

More generally, from the mixed disk amplitude (4.5) with the ADHM constraint (3.30) enforced, we can reconstruct the following anti-hermitian SU(N) connection

$$(\widehat{A}_\mu(x))^u{}_v \equiv -\,\mathrm{i}\, A_\mu(x)^I \, (T^I)^u{}_v = w_{\dot\alpha}{}^u \, (\bar\sigma_{\nu\mu})^{\dot\alpha}{}_{\dot\beta} \, \bar{w}^{\dot\beta}{}_v \, \frac{(x-x_0)^\nu}{(x-x_0)^4} \,, \tag{4.13}$$

which is precisely the leading term in the large distance expansion of the one-instanton connection of the ADHM construction [23] in the singular gauge

$$(\widehat{A}_\mu(x))^u{}_v = w_{\dot\alpha}{}^u \, (\bar\sigma_{\nu\mu})^{\dot\alpha}{}_{\dot\beta} \, \bar{w}^{\dot\beta}{}_v \, \frac{(x-x_0)^\nu}{(x-x_0)^2 \, [(x-x_0)^2 + \rho^2]} \,. \tag{4.14}$$

This analysis clarifies the interpretation of the string amplitude associated to the mixed disk of figure 2. However, a few comments are in order. Firstly, we would like to remark that the amplitude (4.1) is a 3-point function from the point of view of the two dimensional conformal field theory on the string world sheet, but it should be regarded instead as a 1-point function from the point of view of the four-dimensional gauge theory on the D3 branes. Indeed, the two boundary changing operators $V_{\bar{w}}$ and V_w in (4.1) just describe the non-dynamical parameters on which the background depends, *i.e.* the size of the instanton and its orientation inside the gauge group. To emphasize this point, we introduce the convenient notation

$$A^I_\mu(p;\bar{w},w) = \left\langle\!\!\left\langle \mathcal{V}_{A^I_\mu}(-p) \right\rangle\!\!\right\rangle_{\mathcal{D}(\bar{w},w)} \tag{4.15}$$

where $\mathcal{D}(\bar{w},w)$ is the mixed disk produced by the insertion of $V_{\bar{w}}$ and V_w. Secondly, the fact that the instanton connection is in the singular gauge should not come as a surprise, but on the contrary it should be expected in this D-brane set-up. In fact, as we have seen, the gauge instanton is produced by a D(-1) brane which is a point-like object inside the D3 brane world-volume, and thus it is natural that the instanton connection arising in this way exhibits a singularity at the location x_0 of the D-instanton. We recall that in the singular gauge all non-trivial properties of the instanton profile come entirely from the region near the singularity through the embedding of a 3-sphere surrounding x_0 into a SU(2) subgroup of SU(N). This is to be contrasted with what happens in the regular gauge, where all non-trivial properties of the instanton come instead from the asymptotic 3-sphere at infinity. Furthermore, in the singular gauge the instanton field falls off as $1/x^3$ at large distances, thus guaranteeing the convergence of many integrals, like for example that of the topological charge.

An obvious question to ask at this point is whether also the subleading terms in the large distance expansion of the instanton solution can have a direct interpretation in string theory. Since these higher-order terms contain higher powers of ρ^2, they are naturally associated to mixed disks with more insertions of boundary changing operators. For example, the diagram one should consider to study the emission of the vector field at the next-to-leading order is a mixed disk with two more vertices V_w and $V_{\bar{w}}$ as shown in figure 3. However, extending the closed string analysis of ref. [32] to the present case, one can argue that in the limit $\alpha' \to 0$ this diagram reduces to a simpler one in which two first-order diagrams are sewn with a 3-gluon vertex of the SYM theory, as shown in figure 4. In appendix C we will explicitly compute this diagram and find that, for example for SU(2), the corresponding emitted gauge field is

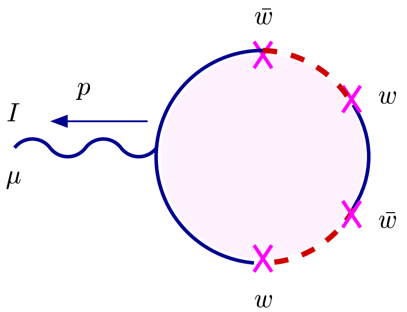

Figure 3: The mixed disk for the second order contribution to the gauge vector.

$$A^c_\mu(x)^{(2)} = -2\rho^4 \bar{\eta}^c_{\mu\nu} \frac{(x-x_0)^\nu}{(x-x_0)^6}\,, \tag{4.16}$$

that is exactly the second-order term in the large distance expansion of $A^c_\mu(x)$ in (4.12). The higher order terms in this expansion can be in principle computed in a similar manner and eventually the full instanton solution can be reconstructed. This analysis shows that the relevant building block for the complete solution is actually the leading term at large distance which corresponds to the “source” diagram of figure 2 whose evaluation, as we have seen, is extremely simple.

What we have described above is the open string analogue of the procedure introduced in refs. [6, 7] for closed strings. There, the so-called boundary states [5, 33] were recognized to be the sources for the various massless fields of the closed string spectrum in a D-brane background, and the classical supergravity D-brane solutions were obtained by taking the Fourier transform of the various tadpoles produced by the boundary states. Similarly here, the mixed disks have been shown to be the sources for the emission of open strings in a background whose profile is precisely that of the classical gauge instanton. Just like the boundary state approach has been very useful to obtain information on the classical geometry associated to complicated D-brane configurations, also the present method based on the use of mixed disks could play a very useful role in determining non-standard classical backgrounds of the gauge theory.

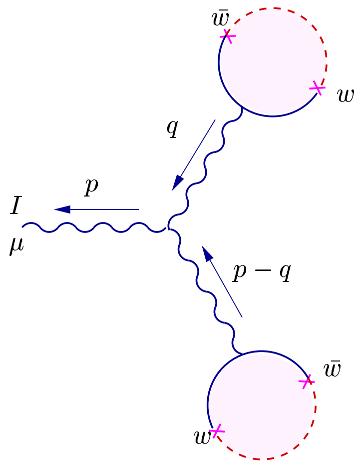

Figure 4: In the field theory limit the mixed disk of figure 3 reduces to this configuration which accounts for the second order term in the large-distance approximation of the instanton solution for the gauge vector.

4.2 Insertions of the translational zero-modes

It is a familiar fact that in the instanton background there are collective coordinates associated to the presence of broken translational symmetries. From the string point of view, these zero-modes describe the motion of the D-instanton within the D3 branes and correspond to the vertex operators of a' (see eq. (2.24)) which, in the 0 superghost picture, are given by

$$V_{a'}^{(0)} = a'_\mu \, \partial_\sigma X^\mu \, . \tag{4.17}$$

These vertex operators can be used to establish in a stringy way a relation between a' and the instanton collective coordinate x_0. Indeed, if one considers all disk diagrams obtained from that of figure 2 by inserting any number of vertices $V_{a'}^{(0)}$ along the D(-1) part of the boundary, and then resums the corresponding perturbative series, one finds that all occurrences of x_0 are replaced by $x_0 + a'$. This fact could be proved by adding to the action of the D(-1) open strings the following marginal deformation along the boundary

$$\delta S = \frac{1}{2\pi\alpha'} \int d\tau \Big[V_{a'}^{(0)}(\sigma = \pi, \tau) - V_{a'}^{(0)}(\sigma = 0, \tau) \Big] \, . \tag{4.18}$$

However, it is quite difficult to treat this interaction in a non-perturbative way, since it is not easy to find an exact solution of the new equations of motion for the string coordinates with the required boundary conditions and regularity properties. For this reason it is convenient to exploit the open-closed string duality and translate the problem into the closed string language. This amounts to represent the D-instanton localized at x_0 with a boundary state $|\mathrm{D}(-1);x_0\rangle$ (see for example ref. [33] for more details) and to perform a world-sheet modular transformation that interchanges the roles of σ and τ. Then, adding the marginal deformation (4.18) to the D(-1) open strings is equivalent, in the closed string channel, to

$$P\,\exp\left(-\frac{\mathrm{i}}{2\pi\alpha'}\int_0^\pi d\sigma\,a'_\mu\,\partial_\tau X^\mu\right)|\mathrm{D}(-1);x_0\rangle\,, \tag{4.19}$$

as one can easily see by generalizing the discussion of ref. [4]. Notice that the path ordering is a consequence of the Chan-Paton factor that must be added to the vertex operator (4.17) when $k>1$. For $k=1$ instead, the path ordering is trivial and the expression (4.19) can be easily evaluated. In particular, one finds that the relevant zero-more part is given by

$$\mathrm{e}^{-\mathrm{i}\,a'_\mu p^\mu}\,\delta^4(x-x_0)\,|p=0\rangle=\delta^4(x-x_0-a')\,|p=0\rangle\,, \tag{4.20}$$

which clearly shows that all occurrences of x_0 are to be replaced by x_0+a', as desired. For this reason in the following we will not distinguish any more between x_0 and a'.

5. The superinstanton profile

The procedure we have discussed in the previous section can be easily extended to the other components of the $\mathcal{N}=4$ vector multiplet, thus allowing to recover the full superinstanton solution from mixed disks. Indeed, acting with the supersymmetry transformations that are preserved also by the D(-1) branes, one can obtain from the diagram of figure 2 those that describe the emission of the gauginos and the scalar fields, and hence their classical profiles as function of the supermoduli. On the other hand, acting with the supersymmetries that are broken by the D(-1) branes, one can shift the supermoduli in the classical solution and account in this way for the fermionic zero-modes of the superinstantons.

5.1 Unbroken supersymmetries

The simplest diagrams which contribute to the emission of a gaugino are mixed disks with one bosonic and one fermionic boundary changing operators. The two possibilities are represented in figure 5. The amplitude (in momentum space) associated to the diagram (a) is given by

$$\bar\Lambda^{\dot\alpha A\,,\,I}(p;\bar w,\mu)\equiv\left\langle\!\!\left\langle\mathcal{V}_{\bar\Lambda^I_{\dot\alpha A}}(-p)\right\rangle\!\!\right\rangle_{\mathcal{D}(\bar w,\mu)}=\left\langle\!\!\left\langle V_{\bar w}^{(-1)}\,\mathcal{V}^{(-1/2)}_{\bar\Lambda^I_{\dot\alpha A}}(-p)\,V_\mu^{(-1/2)}\right\rangle\!\!\right\rangle \tag{5.1}$$

where $\mathcal{D}(\bar w,\mu)$ is the mixed disk created by the insertion of $V_{\bar w}$ and V_μ, and is easily evaluated to be

$$\bar\Lambda^{\dot\alpha A\,,\,I}(p;\bar w,\mu)=\mathrm{i}\,(T^I)^v{}_u\,\mu^{Au}\,\bar w^{\dot\alpha}{}_v\,\mathrm{e}^{-\mathrm{i}p\cdot x_0}\,. \tag{5.2}$$

Notice again that in the amplitude (5.1) we have inserted a gaugino emission vertex with outgoing momentum. Similarly, the amplitude corresponding to the diagram (b) is

$$\bar\Lambda^{\dot\alpha A\,,\,I}(p;\bar\mu,w)\equiv\left\langle\!\!\left\langle\mathcal{V}_{\bar\Lambda^I_{\dot\alpha A}}(-p)\right\rangle\!\!\right\rangle_{\mathcal{D}(w,\bar\mu)}=\mathrm{i}\,(T^I)^v{}_u\,w^{\dot\alpha u}\,\bar\mu^A{}_v\,\mathrm{e}^{-\mathrm{i}p\cdot x_0}\,. \tag{5.3}$$

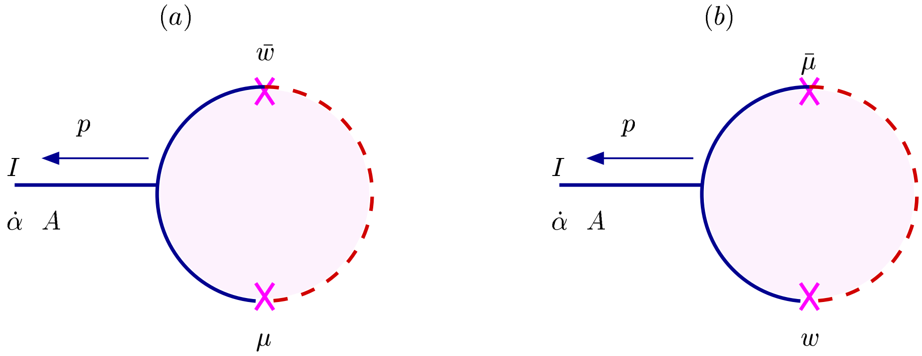

Figure 5: The two mixed disks that contribute to the emission of a gaugino $\bar{\Lambda}^I_{\dot{\alpha}A}$ with momentum p represented by the outgoing solid line.

An alternative method to compute these amplitudes is based on the use of the supersymmetries which are preserved both on the D3 and on the D(-1) boundary and have been denoted by $\bar{\xi}\, q$ in section 2. Exploiting the fact that these supersymmetries annihilate the vacuum, we have the following Ward identity

$$\left\langle\!\!\left\langle \left[\bar{\xi}\, q\,, V_{\bar{w}}\right] \mathcal{V}_{A^I_\mu}(-p)\, V_\mu \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle V_{\bar{w}} \left[\bar{\xi}\, q\,, \mathcal{V}_{A^I_\mu}(-p)\right] V_\mu \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle V_{\bar{w}}\, \mathcal{V}_{A^I_\mu}(-p) \left[\bar{\xi}\, q\,, V_\mu\right] \right\rangle\!\!\right\rangle = 0\,, \quad (5.4)$$

where for simplicity we have understood the picture assignments.[8] The only new ingredient appearing in (5.4) is the commutator in the second term; this can be computed from (4.2) and reads

$$\left[\bar{\xi}\, q\,, \mathcal{V}_{A^I_\mu}(-p)\right] = \bar{\xi}_{\dot{\beta}A}\, p_\nu\, (\bar{\sigma}^{\nu\mu})^{\dot{\beta}}{}_{\dot{\alpha}}\, \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p)\,. \quad (5.5)$$

Then, using (2.36) and (2.37), we can rewrite the Ward identity (5.4) as follows

$$\bar{\xi}_{\dot{\beta}A}\, p_\nu\, (\bar{\sigma}^{\nu\mu})^{\dot{\beta}}{}_{\dot{\alpha}} \left\langle\!\!\left\langle V_{\bar{w}}\, \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p)\, V_\mu \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle V_{\bar{w}}\, \mathcal{V}_{A^I_\mu}(-p)\, V_{\delta_{\bar{\xi}} w} \right\rangle\!\!\right\rangle = 0 \quad (5.6)$$

which allows to obtain the gaugino amplitude in terms of the gauge boson amplitude (4.4) with w replaced by its supersymmetry variation $\delta_{\bar{\xi}} w$ given in (2.34). In this way we can immediately get (5.2), and with a similar relation also (5.3) can be retrieved.

The space-time profile of the emitted gaugino is then obtained by taking the Fourier transform of the sum of the amplitudes (5.2) and (5.3) multiplied by the free fermion propagator $\mathrm{i}\, \not{p}^{\dot{\beta}\alpha}/p^2 \equiv \mathrm{i}\, p^\nu (\bar{\sigma}_\nu)^{\dot{\beta}\alpha}/p^2$, that is

$$\begin{aligned} \Lambda^{\alpha A\,,\,I}(x) &= \int \frac{d^4p}{(2\pi)^2} \left(\bar{\Lambda}_{\dot{\beta}}{}^{A\,,\,I}(p;\bar{w},\mu) + \bar{\Lambda}_{\dot{\beta}}{}^{A\,,\,I}(p;\bar{\mu},w)\right) \frac{\mathrm{i}\, \not{p}^{\dot{\beta}\alpha}}{p^2}\, \mathrm{e}^{\mathrm{i}p\cdot x} \\ &= -2\mathrm{i}\, (T^I)^v{}_u \left(w_{\dot{\beta}}{}^u\, \bar{\mu}^A{}_v + \mu^{Au}\, \bar{w}_{\dot{\beta}v}\right) (\bar{\sigma}_\nu)^{\dot{\beta}\alpha}\, \frac{(x-x_0)^\nu}{(x-x_0)^4}\,. \end{aligned} \quad (5.7)$$

Just as the gauge field (4.10), also the gaugino (5.7) naturally arises in terms of *unconstrained* parameters which become the instanton moduli when they are restricted to satisfy

[8] The latter are $(-1/2)$, 0 and (-1) for V_μ, $\mathcal{V}_{A^I_\mu}$ and $V_{\bar{w}}$ respectively, and $(-1/2)$ for the supercharges.

JHEP02(2003)045

the ADHM constraints (3.30) and (3.31). In particular, once the fermionic constraint (3.31) is imposed, it is immediate to extract from (5.7) the following matrix-valued gaugino profile

$$(\widehat{\Lambda}^{\alpha A}(x))^u{}_v \equiv -\,\mathrm{i}\,\Lambda^{\alpha A\,,I}(x)\,(T^I)^u{}_v = (\sigma_\nu)^\alpha{}_{\dot\beta}\left(w^{\dot\beta u}\,\bar\mu^A{}_v + \mu^{Au}\,\bar w^{\dot\beta}{}_v\right)\frac{(x-x_0)^\nu}{(x-x_0)^4}\,. \tag{5.8}$$

In this expression we recognize exactly the leading term in the large distance expansion of the gaugino instanton solution in the singular gauge (see for example appendix B)

$$(\widehat{\Lambda}^{\alpha A}(x))^u{}_v = (\sigma_\nu)^\alpha{}_{\dot\beta}\left(w^{\dot\beta u}\,\bar\mu^A{}_v + \mu^{Au}\,\bar w^{\dot\beta}{}_v\right)\frac{(x-x_0)^\nu}{\sqrt{(x-x_0)^2\Big[(x-x_0)^2+\rho^2\Big]^3}}\,. \tag{5.9}$$

The subleading terms can be obtained from diagrams with more sources, in complete analogy with what we did for the gauge field.

Let us now turn to the scalar components φ^I_a of the $\mathcal{N}=4$ vector multiplet. The simplest diagram which can describe their emission is a mixed disk with two fermionic boundary changing operators, like the one represented in figure 6. The corresponding amplitude in momentum space is

$$\begin{aligned}\varphi^I_a(p;\bar\mu,\mu) &\equiv \Big\langle\!\!\Big\langle \mathcal{V}_{\varphi^I_a}(-p)\Big\rangle\!\!\Big\rangle_{\mathcal{D}(\bar\mu,\mu)}\\ &= \Big\langle\!\!\Big\langle V^{(-1/2)}_{\bar\mu}\,\mathcal{V}^{(-1)}_{\varphi^I_a}(-p)\,V^{(-1/2)}_{\mu}\Big\rangle\!\!\Big\rangle\\ &= -\,\frac{\mathrm{i}}{2}\,(T^I)^v{}_u\,(\bar\Sigma_a)_{AB}\,\mu^{Bu}\,\bar\mu^A{}_v\,\mathrm{e}^{-\mathrm{i}p\cdot x_0}\end{aligned} \tag{5.10}$$

where $\mathcal{D}(\bar\mu,\mu)$ is the mixed disk created by the insertion of $V_{\bar\mu}$ and V_μ. Defining

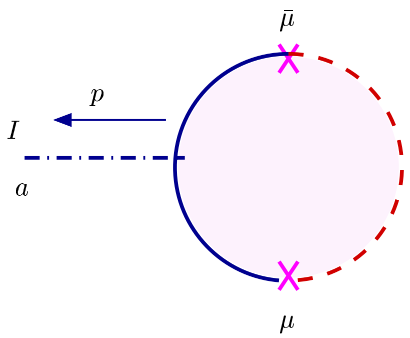

Figure 6: The mixed disk describing the emission of an adjoint scalar φ^I_a of momentum p represented by the outgoing dashed line.

$$\varphi^{AB} = \frac{1}{2\sqrt2}\,(\Sigma^a)^{AB}\,\varphi^a\,, \tag{5.11}$$

we can rewrite (5.10) as

$$\varphi^{AB\,,I}(p;\bar\mu,\mu) = -\,\frac{\mathrm{i}}{\sqrt2}\,(T^I)^v{}_u\,\mu^{[Au}\,\bar\mu^{B]}{}_v\,\mathrm{e}^{-\mathrm{i}p\cdot x_0} \tag{5.12}$$

where the square brackets mean antisymmetrization with weight one. Alternatively, this result can be obtained from the Ward identity

$$\Big\langle\!\!\Big\langle \Big[\bar\xi\,q\,,V_{\bar\mu}\Big]\,\mathcal{V}_{\bar\Lambda^I_{\dot\alpha A}}(-p)\,V_\mu\Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle V_{\bar\mu}\Big[\bar\xi\,q\,,\mathcal{V}_{\bar\Lambda^I_{\dot\alpha A}}(-p)\Big]\,V_\mu\Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle V_{\bar\mu}\,\mathcal{V}_{\bar\Lambda^I_{\dot\alpha A}}(-p)\,\Big[\bar\xi\,q\,,V_\mu\Big]\Big\rangle\!\!\Big\rangle = 0 \tag{5.13}$$

which establishes a relation between the scalar and the gaugino amplitudes.[9] Indeed, working out the commutators, we find

$$\Big\langle\!\!\Big\langle V_{\delta_{\bar\xi}\bar w}\,\mathcal{V}_{\bar\Lambda^I_{\dot\alpha A}}(-p)\,V_\mu\Big\rangle\!\!\Big\rangle - \mathrm{i}\,\bar\xi^{\dot\alpha}_B\,(\Sigma^a)^{BA}\Big\langle\!\!\Big\langle V_{\bar\mu}\,\mathcal{V}_{\varphi^I_a}(-p)\,V_\mu\Big\rangle\!\!\Big\rangle + \Big\langle\!\!\Big\langle V_{\bar\mu}\,\mathcal{V}_{\bar\Lambda^I_{\dot\alpha A}}(-p)\,V_{\delta_{\bar\xi}w}\Big\rangle\!\!\Big\rangle = 0\,, \tag{5.14}$$

from which (5.12) easily follows upon using (5.2), (5.3) and (2.34).

[9] In eq. (5.13) all vertex operators, as well as the supersymmetry charges, are in the $(-1/2)$ picture.

The space-time profile of the adjoint scalars is obtained by taking the Fourier transform of the amplitude (5.12) multiplied by the massless scalar propagator $1/p^2$, namely

$$\begin{aligned}\varphi^{AB\,,I}(x) &= \int \frac{d^4p}{(2\pi)^2}\,\varphi^{AB\,,I}(p;\bar{\mu},\mu)\,\frac{1}{p^2}\,\mathrm{e}^{\mathrm{i}p\cdot x}\\ &= -\,\frac{\mathrm{i}}{\sqrt{2}}\,(T^I)^v{}_u\,\mu^{[Au}\,\bar{\mu}^{B]}_{\ v}\,\frac{1}{(x-x_0)^2}\,.\end{aligned} \tag{5.15}$$

When the parameters are restricted to satisfy the ADHM constraints, this expression represents the leading term of the adjoint scalars in the singular gauge. Moreover, from (5.15) one can see that

$$(\widehat{\varphi}^{AB}(x))^u{}_v \equiv -\,\mathrm{i}\,\varphi^{AB\,,I}(x)\,(T^I)^u{}_v = -\,\frac{1}{2\sqrt{2}}\left(\mu^{[Au}\,\bar{\mu}^{B]}_{\ v} - \frac{1}{2}\,\mu^{[Ap}\,\bar{\mu}^{B]}_{\ p}\,\tilde{\delta}^u{}_v\right)\frac{1}{(x-x_0)^2} \tag{5.16}$$

with

$$\|\tilde{\delta}^u{}_v\| = \begin{pmatrix} 0_{[N-2]\times[N-2]} & 0_{[N-2]\times[2]}\\ 0_{[2]\times[N-2]} & 1_{[2]\times[2]}\end{pmatrix}\,, \tag{5.17}$$

which is indeed the leading term at large distance of the exact instanton solution (see for example appendix B). As before, the subleading terms are given by diagrams with more insertions of source terms.

We can summarize our findings by saying that the mixed disks with two boundary changing operators represented in figures 2, 5 and 6 describe, respectively, the large distance behavior in the instanton background of the vector A^I_μ, of the gaugino $\Lambda^I_{\alpha A}$ and of the scalars φ^I_{AB} in the singular gauge, and that their space-time profiles can be written as

$$\begin{aligned}A^I_\mu(x) &= J^I_{\nu\mu}\,\partial^\nu G(x-x_0)\,,\\ \Lambda^{\alpha A\,,I}(x) &= J^{A\,,I}_{\dot\beta}\,(\bar{\sigma}^\nu)^{\dot\beta\alpha}\,\partial_\nu G(x-x_0)\,,\\ \varphi^{AB\,,I}(x) &= J^{AB\,,I}\,G(x-x_0)\,,\end{aligned} \tag{5.18}$$

where the scalar Green function $G(x-x_0)$ is defined in (4.7) and the various source terms $J^I_{\nu\mu}$, $J^{A\,,I}_{\dot\beta}$ and $J^{AB\,,I}$ are bilinear expressions in the instanton moduli which can be read from (4.4), (5.2), (5.3) and (5.10) respectively. Moreover, taking into account the fall-off at infinity of the various fields, one can easily realize that the equations of motion that follow from the SYM action (3.11) in the Lorentz gauge reduce at large distances simply to free equations,[10] i.e.

$$\Box A^I_\mu = 0\,, \qquad \bar{\partial\!\!\!/}_{\alpha\dot\beta}\,\Lambda^{\alpha A\,,I} = 0\,, \qquad \Box\varphi^{AB\,,I} = 0\,, \tag{5.19}$$

which indeed admit a solution of the form (5.18) in the presence of source terms.

[10] At first sight there is a problem with the large distnace behaviour of $\varphi\partial\varphi \sim O(1/x^5)$ which is the same (and not a stronger) behaviour of $\partial^2 A$. However the eq.s (5.19) are still valid since the dangerous terms in A^{AB} equation of motion, namely $\left[\bar{\varphi}_{AB}, \partial_\mu\varphi^{AB}\right]$, cancel identically: upon use of eq. (5.18) they become porportional to $\epsilon_{ABCD}[J^{AB}, J^{CD}] = 0$.

5.2 Broken supersymmetries

Let us now consider the supersymmetries of the D3 branes which are broken by the D-instantons, namely those that are generated by the charges $q'_{\alpha A} \equiv \left(Q_{\alpha A} + \widetilde{Q}_{\alpha A}\right)$ (see section 2.1). As shown in (2.9), when one pulls the integration contour of a charge operator to a boundary that does not preserve it, one obtains the integrated emission vertex for the Goldstone field corresponding to the broken charge. In our case, the goldstino associated to the breaking of $q'_{\alpha A}$ by the D(-1) boundary is the modulus $M'^{\alpha A}$. Therefore, by acting with the broken supercharges $q'_{\alpha A}$ on a given instanton solution, one can modify it by shifting its supermoduli with M' dependent terms. In particular, one can relate the "minimal" emission diagrams of figures 2, 5 and 6, that contain no D(-1)/D(-1) moduli, to diagrams which instead have additional insertions of M' moduli [18]. Thus, the use of the broken supersymmetries allows us to determine the M' dependence and complete the full superinstanton solution.

Let us see how this works in a specific example and consider the following Ward identity

$$\left\langle\!\!\left\langle \Big[M'q', V_{\bar{w}}\Big] \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p) V_w \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle V_{\bar{w}} \Big[M'q', \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p)\Big] V_w \right\rangle\!\!\right\rangle + \tag{5.20}$$
$$+ \left\langle\!\!\left\langle V_{\bar{w}} \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p) \Big[M'q', V_w\Big] \right\rangle\!\!\right\rangle = -\left\langle\!\!\left\langle V_{\bar{w}} \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p) V_w \int V_{M'} \right\rangle\!\!\right\rangle .$$

Differently from the identities (5.4) and (5.13) associated to the preserved supersymmetries, the right hand side of (5.20) is non-zero as a consequence of the fact that the supercharge q' is broken on the D(-1) boundary. A pictorial representation of this Ward identity is provided in figure 7. Using the fact that the commutators of q' with V_w and $V_{\bar{w}}$ vanish (as we already noticed at the end of section 2), and that

$$\Big[M'q', \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p)\Big] = \mathrm{i}\, M'^{\beta A}\, (\sigma_\mu)_\beta^{\ \dot{\alpha}}\, \mathcal{V}_{A^I_\mu}(-p)\,, \tag{5.21}$$

we can deduce from (5.20) the following relation

$$\bar{\Lambda}^{\dot{\alpha}A,I}(p; \bar{w}, w, M') \equiv \left\langle\!\!\left\langle \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p) \right\rangle\!\!\right\rangle_{\mathcal{D}(\bar{w},w,M')} = \left\langle\!\!\left\langle V_{\bar{w}}\, \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}(-p)\, V_w \int V_{M'} \right\rangle\!\!\right\rangle -$$
$$= -\,\mathrm{i}\, M'^{\beta A}\, (\sigma^\mu)_\beta^{\ \dot{\alpha}}\, A^I_\mu(p; \bar{w}, w)\,, \tag{5.22}$$

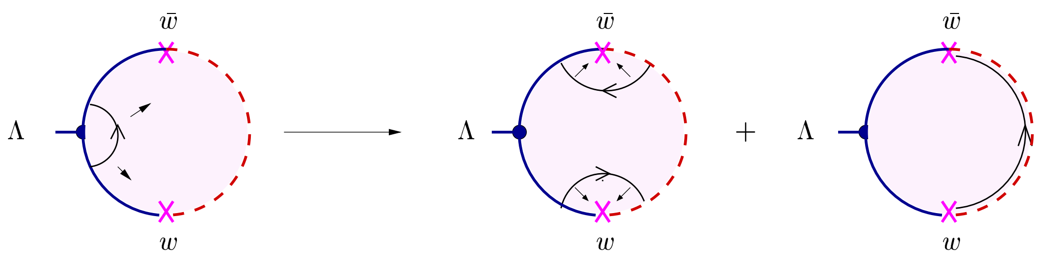

Figure 7: The Ward identity for the broken supersymmetries. The internal oriented line represents the integration contour for the supercurrent $M'(j+\tilde{\jmath})$. The diagram in the left hand side corresponds to the term $\langle\!\langle V_{\bar{w}}[M'q', \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}}] V_w \rangle\!\rangle$ in (5.20). The two diagrams in the right hand side are obtained by deforming the integration contour. The first of them corresponds to $-\langle\!\langle [M'q', V_{\bar{w}}] \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}} V_w \rangle\!\rangle - \langle\!\langle V_{\bar{w}}\, \mathcal{V}_{\bar{\Lambda}^I_{\dot{\alpha}A}} [M'q', V_w] \rangle\!\rangle$ (where the minus sign is due to the clockwise orientation of the contours), whereas the last diagram corresponds to the right hand side of (5.20).

JHEP02(2003)045

which reduces the calculation of the 4-point amplitude $\bar{\Lambda}^{\dot{\alpha}A\,,\,I}(p;\bar{w},w,M')$ to an algebraic manipulation on the 3-point amplitude (4.1). Notice again that, despite the presence of many vertex operators, the amplitude (5.22) is actually a 1-point function from the point of view of the four-dimensional gauge theory, since the only dynamical field is the emitted gaugino. To obtain its corresponding space-time profile we multiply $\bar{\Lambda}_{\dot{\beta}}^{\;\dot{\alpha}A\,,\,I}(p;w,\bar{w},M')$ by the propagator $\mathrm{i}\,\not{p}^{\dot{\beta}\alpha}/p^2$ and take the Fourier transform, getting

$$
\begin{aligned}
\Lambda^{\alpha A\,,\,I}(x) &= \int \frac{d^4p}{(2\pi)^2}\,\bar{\Lambda}_{\dot{\beta}}^{\;A\,,\,I}(p;\bar{w},w,M')\,\frac{\mathrm{i}\,\not{p}^{\dot{\beta}\alpha}}{p^2}\,\mathrm{e}^{\mathrm{i}p\cdot x} \\
&= M'^{\beta A}\,(\sigma^\mu\,\bar{\sigma}^\nu)_\beta^{\;\alpha}\int\frac{d^4p}{(2\pi)^2}\,\frac{p_\nu\,A^I_\mu(p;w,\bar{w})}{p^2}\,\mathrm{e}^{\mathrm{i}p\cdot x} \\
&= -\,\mathrm{i}\,M'^{\beta A}\,(\sigma^\mu\,\bar{\sigma}^\nu)_\beta^{\;\alpha}\,\partial_\nu\,A^I_\mu(x) \;\stackrel{x\to\infty}{\simeq}\; \frac{\mathrm{i}}{2}\,M'^{\beta A}\,(\sigma^{\mu\nu})_\beta^{\;\alpha}\,F^I_{\mu\nu}(x)\,.
\end{aligned}
\tag{5.23}
$$

In the last step we have used the fact that in the instanton solution (4.4) the vector field A^I_μ is in the Lorenz gauge and that, due to the fall-off at infinity of the potential, the associated non-abelian field strength $F^I_{\mu\nu}$ simply reduces to $\partial_\mu A^I_\nu - \partial_\nu A^I_\mu$ in the large distance limit. Eq. (5.23) shows that a mixed disk with one M' insertion and one emitted gaugino reproduces exactly the chiral fermionic profile that is created by acting with a broken supercharge on the instanton background according to the η-supersymmetry transformation rules (2.20). Of course, with a repeated use of these supercharges, further insertions of M' can be obtained and the entire structure of the superinstanton zero-modes can be reconstructed (see for example eq. (4.60) in the recent review [22]). Our analysis, which for simplicity we have illustrated only in the simplest case, shows the precise relation between these zero-modes and the mixed disk amplitudes with insertions of M' vertex operators. Finally, we recall that with the replacement

$$
M'^{\alpha A} \to -\bar{\zeta}_{\dot{\alpha}}^{\;A}\,(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\,a'_\mu \tag{5.24}
$$

one can account for the superconformal zero-modes of the $\mathcal{N}=4$ instanton solution parametrized by the fermionic variables $\bar{\zeta}$.

6. String amplitudes and instanton calculus

In this section we want to explain what is the stringy procedure to compute instanton corrections to scattering amplitudes in gauge theories and show its relation with the standard instanton calculus of field theory. The key ingredient will be the identification of the instanton solution with the string theory 1-point function on mixed disks that we have proven in the previous sections. Exploiting this fact, we will also be able to relate our approach to the analysis of the leading D-instantons effects on scattering amplitudes that has been presented in ref. [18]. Let us first recall a few basic facts on the relation between string theory correlators, effective actions and Green functions in field theory. As we have reviewed in section 3, the tree-level scattering amplitude among n states of the 3/3 strings

JHEP02(2003)045

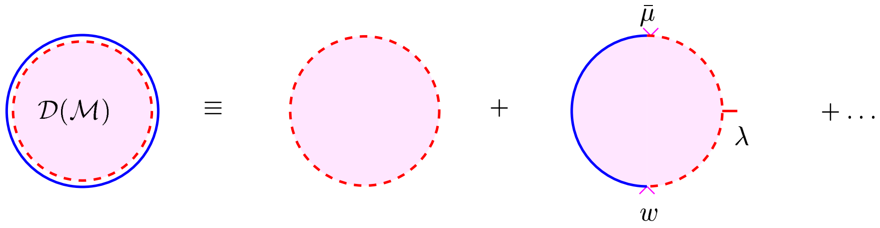

Figure 8: Pictorial representation of the "disk" $\mathcal{D}(\mathcal{M})$. For example, the second disk in the r.h.s. corresponds to the amplitude $\mathcal{A}_{(w\lambda\bar\mu)}$ (see eq. (3.22)) which in the field theory limit gives rise to the term $\mathrm{tr}\left(\mathrm{i}\,\bar\mu^A{}_u w_{\dot\alpha}{}^u \lambda^{\dot\alpha}{}_A\right)$ of the moduli action.

(which we denote generically by ϕ_i) is given by[11]

$$\mathcal{A}_{\phi_1\cdots\phi_n} = \left\langle\!\!\left\langle V_{\phi_1}(p_1)\cdots V_{\phi_n}(p_n)\right\rangle\!\!\right\rangle \equiv \phi_n(p_n)\cdots\phi_1(p_1)\left\langle\!\!\left\langle \mathcal{V}_{\phi_1}(p_1)\cdots \mathcal{V}_{\phi_n}(p_n)\right\rangle\!\!\right\rangle \tag{6.1}$$

where the correlator among the vertex operators is computed on a disk with D3 boundary conditions (see for example eq. (3.1)). By taking the limit $\alpha'\to 0$ and extracting the 1PI part, we obtain the following contribution to the effective action

$$-\int \frac{d^4p_1}{(2\pi)^2}\cdots\frac{d^4p_n}{(2\pi)^2}\phi_n(p_n)\cdots\phi_1(p_1)\left\langle\!\!\left\langle \mathcal{V}_{\phi_1}(p_1)\cdots \mathcal{V}_{\phi_n}(p_n)\right\rangle\!\!\right\rangle\Big|^{\mathrm{1PI}}_{\alpha'\to 0}\,, \tag{6.2}$$

which, in turn, induces the following *amputated* Green function[12]

$$\left\langle \phi_1(p_1)\cdots\phi_n(p_n)\right\rangle\Big|_{\mathrm{amput.}} = \left\langle\!\!\left\langle \mathcal{V}_{\phi_1}(-p_1)\cdots \mathcal{V}_{\phi_n}(-p_n)\right\rangle\!\!\right\rangle\Big|^{\mathrm{1PI}}_{\alpha'\to 0}\,. \tag{6.3}$$

If one computes the above correlators on world-sheets with more boundaries one obtains the perturbative loop corrections to the effective action and Green functions.

We now want to investigate how the previous relations get modified by the presence of k D-instantons. In this case, as we have thoroughly explained, the correlators of vertex operators receive contributions also from world-sheets with a part of their boundary on the D-instantons, and specifically, at the lowest order in the string perturbation theory, from mixed disks. It is convenient to denote by $\mathcal{D}(\mathcal{M})$ the sum of all disks with all possible insertions of the moduli $\mathcal{M}$ of the k instantons, as represented in figure 8. Each term in this sum corresponds to an amplitude with no vertex operator of the 3/3 strings, and thus it represents a vacuum contribution from the point of view of the theory on the D3 branes. A noteworthy point is that also the first term in $\mathcal{D}(\mathcal{M})$, i.e. the pure D(-1) disk without insertions, contributes. Indeed, as shown in ref. [10], it evaluates to minus k times the topological normalization C_0 given in (3.8). Collecting all terms and using the results of section 3, we obtain that the vacuum contribution of the "disk" $\mathcal{D}(\mathcal{M})$ is such that

$$\langle\!\langle 1\rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \stackrel{\alpha'\to 0}{\simeq} -\,S[\mathcal{M}] \equiv -\frac{8\pi^2 k}{g^2_{\mathrm{YM}}} - S_{\mathrm{moduli}} \tag{6.4}$$

where the moduli action is defined in (3.29).

[11] Suitable symmetry factors must be included when not all field ϕ_i are different.

[12] For simplicity, we assume that the propagators are $\langle\phi_i(p)\phi_j(k)\rangle = (2\pi)^2\delta^4(p+k)\,\frac{\delta_{ij}}{p^2}$; if this does not happen, like for instance for the gauginos, appropriate changes are required, but these can be straightforwardly implemented in our formulas.

JHEP02(2003)045

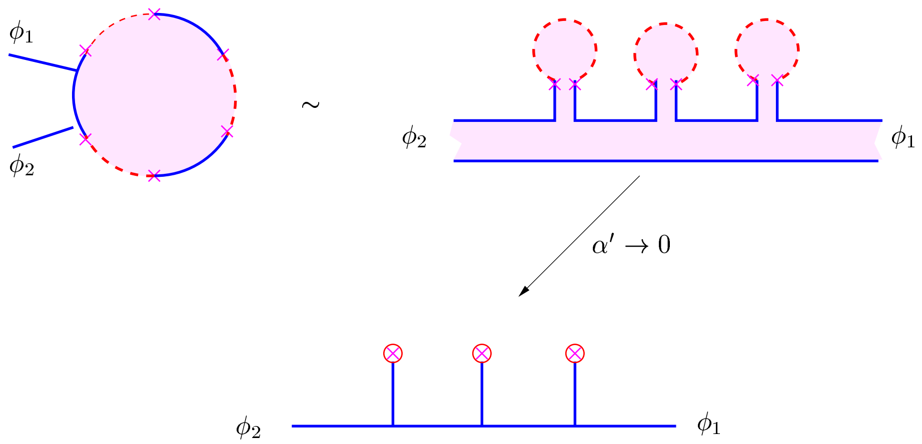

Figure 9: Mixed disk string diagrams correspond in the field-theory limit to interactions of the first-quantized world-lines with the instantonic background.

Let us now consider the correlators of 3/3 string vertex operators on $\mathcal{D}(\mathcal{M})$, which are defined by

$$\left\langle\!\!\left\langle \mathcal{V}_{\phi_1}(p_1)\ldots\mathcal{V}_{\phi_n}(p_n)\right\rangle\!\!\right\rangle_{\mathcal{D}(\mathcal{M})} = \tag{6.5}$$

$$= C_0\sum_m\int\frac{\prod_i dz_i\,\prod_j dy_j}{dV_{abc}}\,\left\langle\mathcal{V}_{\phi_1}(z_1;p_1)\cdots\mathcal{V}_{\phi_n}(z_n;p_n)V_{\mathcal{M}_1}(y_1)\cdots V_{\mathcal{M}_m}(y_m)\right\rangle .$$

As is obvious from this definition, the string theory correlator depends on the k-instanton moduli $\mathcal{M}$, over which one has to integrate in order to account for all possible configurations. This fact is intuitively clear, since in our description all possible mixed boundary conditions are obtained by inserting the moduli.

The integration over $\mathcal{M}$ is the analogue of what one typically does in quantum field theory, where the path integral describing a specific correlator is split into the sum of path integrals restricted to the different topological sectors, namely

$$\int\mathcal{D}\phi\,\phi_1(p_1)\cdots\phi_n(p_n)\,\mathrm{e}^{-S[\phi]} = \sum_k\int\mathcal{D}\delta\phi^{(k)}\,\delta\phi_1^{(k)}(p_1)\cdots\delta\phi_n^{(k)}(p_n)\,\mathrm{e}^{-S_k-S[\delta\phi^{(k)}]} \tag{6.6}$$

where $\delta\phi^{(k)}$ denotes the fluctuation of ϕ around a classical background with topological charge k and action S_k. In this framework, the integration over all moduli of the non-trivial background arises directly from the path-integral, as a trade-off for the integration over the zero-mode fluctuations. However, from string theory we obtain a first-quantized description in which the string world-sheet gives rise for $\alpha' \to 0$ to the world-lines of a (super)particle description of the Feynman diagrams of the field theory. In this description, the different topological sectors can be described only by explicitly coupling the (super)particle to a non-trivial background field A_μ through the insertion of

$$\mathrm{Tr}\,P\exp\left(\int_\gamma A_\mu(x(\tau);\mathcal{M})\dot{x}^\mu\,d\tau\right) \tag{6.7}$$

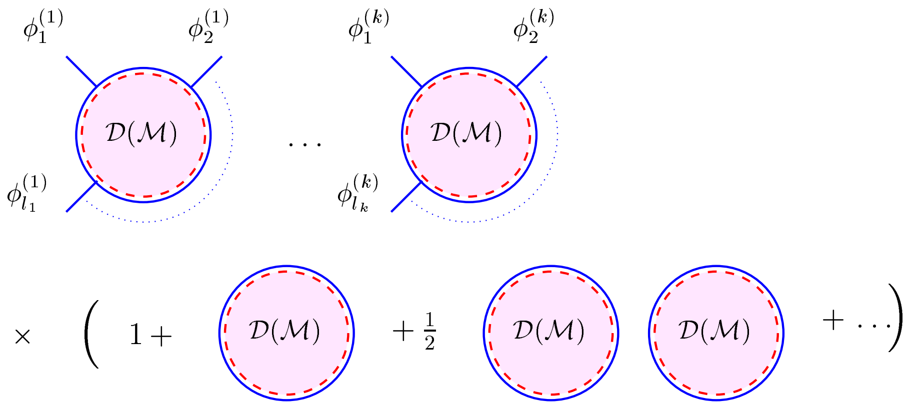

Figure 10: A connected amplitude with n external $\mathcal{V}_\phi$ vertex operators in a D-instanton background receives contributions from topologically disconnected world-sheets, characterized by the insertion of l_i vertex operators for ϕ fields in each connected component, with $\sum_i l_i = n$.

and then integrating over the background parameters $\mathcal{M}$. This procedure is pictorially illustrated in figure 9 for a specific disk amplitude.

The integration over the moduli $\mathcal{M}$ has several important consequences. First of all, also world-sheets with disconnected components must be taken into account. For example, besides the correlator (6.5), one should also consider the following one

$$\langle\!\langle \mathcal{V}_{\phi_1}(p_1) \cdots \mathcal{V}_{\phi_n}(p_n) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \langle\!\langle 1 \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \, , \tag{6.8}$$

which is disconnected from the two-dimensional point of view but connected from the point of view of the four-dimensional theory on the D3 branes. Obviously, we can add more disconnected components, and thus in general we have

$$\frac{1}{\ell\,!} \langle\!\langle \mathcal{V}_{\phi_1}(p_1) \cdots \mathcal{V}_{\phi_n}(p_n) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \, \left(\langle\!\langle 1 \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \right)^\ell \tag{6.9}$$

where the symmetry factor is due to the combinatorics of boundaries [10]. Summing over all these terms, we therefore get

$$\langle\!\langle \mathcal{V}_{\phi_1}(p_1) \cdots \mathcal{V}_{\phi_n}(p_n) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \, \mathrm{e}^{\langle\!\langle 1 \rangle\!\rangle_{\mathcal{D}(\mathcal{M})}} \, . \tag{6.10}$$

However, this is not yet the full story. In fact, for the same arguments we should also take into account diagrams in which the n vertex opertors $\mathcal{V}_{\phi_i}(p_i)$ are distributed among various disconnected components. For example, besides the correlator (6.10) we should also consider the following one

$$\langle\!\langle \mathcal{V}_{\phi_1}(p_1)\, \mathcal{V}_{\phi_2}(p_2) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \, \langle\!\langle \mathcal{V}_{\phi_3}(p_3) \cdots \mathcal{V}_{\phi_n}(p_n) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \, \mathrm{e}^{\langle\!\langle 1 \rangle\!\rangle_{\mathcal{D}(\mathcal{M})}} \, . \tag{6.11}$$

This contribution appears to be totally disconnected; however, it is connected with respect to the ϕ's because of the integration over the moduli $\mathcal{M}$ which all sit at the same point where the stack of k D-instantons is located. Distributing the ϕ's in all possible ways, one generates various configurations which are compactly represented in figure 10.

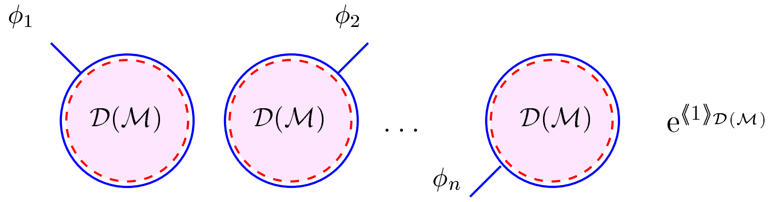

Figure 11: The dominant contribution to an amplitude with n external $\mathcal{V}_\phi$ vertex operators in a D-instanton background is a product of tadpoles.

Since each expectation value on $\mathcal{D}(\mathcal{M})$ is proportional to $C_0 \propto g_s^{-1}$ (see eqs. (6.5) and (3.8)), the dominant contribution for small g_s is the one in which a single vertex $\mathcal{V}_\phi$ is inserted in each disk [11, 18], namely

$$\langle\!\langle \mathcal{V}_{\phi_1}(p_1) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \cdots \langle\!\langle \mathcal{V}_{\phi_n}(p_n) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})}\, \mathrm{e}^{\langle\!\langle\, 1 \,\rangle\!\rangle_{\mathcal{D}(\mathcal{M})}}\,, \tag{6.12}$$

whereas other terms, like for example (6.11), are subleading for small g_s.[13] Moreover, this correlator is clearly 1PI. Thus, we can conclude that in the field theory limit, the dominant contribution to the amputated Green function of n fields of the 3/3 string sector in the presence of k D-instanton is given by (see figure 11)

$$\begin{aligned}&\langle \phi_1(p_1) \ldots \phi_n(p_n) \rangle \big|^{\mathrm{D-inst.}}_{\mathrm{amput.}} = \\ &\quad = \int d\mathcal{M}\; \langle\!\langle \mathcal{V}_{\phi_1}(-p_1) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \cdots \langle\!\langle \mathcal{V}_{\phi_n}(-p_n) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})}\, \mathrm{e}^{\langle\!\langle 1 \rangle\!\rangle_{\mathcal{D}(\mathcal{M})}} \Big|_{\alpha' \to 0}\,.\end{aligned} \tag{6.13}$$

Reinstating the propagators (see footnote 12) and Fourier transforming, we obtain the following Green function in configuration space

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \big|_{\mathrm{D-inst.}} = \int d\mathcal{M}\; \phi_1^{\mathrm{disk}}(x_1; \mathcal{M}) \cdots \phi_n^{\mathrm{disk}}(x_n; \mathcal{M})\, \mathrm{e}^{-S[\mathcal{M}]} \tag{6.14}$$

where we have used (6.4) and defined

$$\phi^{\mathrm{disk}}(x; \mathcal{M}) = \int \frac{d^4p}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}p\cdot x}\, \frac{1}{p^2}\, \langle\!\langle \mathcal{V}_\phi(-p) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \big|_{\alpha' \to 0}\,. \tag{6.15}$$

Using the results of sections 4 and 5 we can identify the right hand side of (6.15) with the classical profile $\phi^{\mathrm{cl}}(x; \mathcal{M})$ of the superinstanton solution for the field ϕ. For example, the contributions from the simplest mixed disks, i.e. those with only two insertions of boundary changing operators, account for the leading terms in the large distance expansion of the superinstanton solution, as we have seen explicitly for $k = 1$ in eqs. (4.5), (5.7) and (5.15). The contributions from mixed disks with more boundary changing operators in the limit $\alpha' \to 0$ account instead for the sub-leading terms in the large distance expansion, as we have shown for the gauge field in section 4 (see also appendix C). Thus, we can write

$$\phi(x; \mathcal{M})^{\mathrm{disk}} = \phi^{\mathrm{cl}}(x; \mathcal{M}) \tag{6.16}$$

[13] Notice that world-sheets with higher Euler number can also give contributions to the sub-leading orders.

and conclude that the stringy prescription (6.14) of computing correlation functions in the presence of D-instantons is exactly equivalent to the standard field theory prescription of the instanton calculus

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle \big|_{\text{inst.}} = \int d\mathcal{M}\; \phi_1^{\text{cl}}(x_1;\mathcal{M}) \cdots \phi_n^{\text{cl}}(x_n;\mathcal{M})\, \mathrm{e}^{-S[\mathcal{M}]}\,. \tag{6.17}$$

The effects of D-instantons on the scattering amplitudes of the gauge theory on the D3 branes can be encoded by introducing new effective vertices for the 3/3 fields ϕ_i's which suitably modify the SYM action (see also ref. [18]). These D-instanton induced vertices originate from the amputated Green functions (6.13) upon including the polarization fields for the external legs, and are clearly moduli dependent. At *fixed* moduli, only the 1-point functions are irreducible and so the gauge effective action induced by the D-instantons on the D3 branes will be

$$S_{(-1)/3} = -\sum_{\phi} \int \frac{d^4p}{(2\pi)^2}\, \phi(p)\, \langle\!\langle \mathcal{V}_\phi(p) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})} \big|_{\alpha' \to 0} \tag{6.18}$$

where the sum is over all massless fields of the $\mathcal{N} = 4$ vector multiplet. Since the tadpoles $\langle\!\langle \mathcal{V}_\phi(p) \rangle\!\rangle_{\mathcal{D}(\mathcal{M})}$ are generically of the form $J_\phi(\mathcal{M})\, \mathrm{e}^{\mathrm{i}p\cdot x_0}$ (see for instance eqs. (5.2) and (5.10)),[14] we can write this effective action simply as

$$S_{(-1)/3} = -\sum_{\phi} \phi(x_0)\, J_\phi(\mathcal{M}) \tag{6.19}$$

which manifestly shows that the 1-point functions on the mixed disks are sources for the gauge fields at the instanton location. Using the expressions for the various tadpoles computed in sections 4 and 5, it is easy to realize that

$$S_{(-1)/3} = -\frac{1}{2}\, F^I_{\mu\nu}(x_0)\, J^{\mu\nu\,,I}(\mathcal{M}) - \bar{\Lambda}^I_{\dot{\alpha}A}(x_0)\, J^{\dot{\alpha}A\,,I}(\mathcal{M}) - \varphi^I_{AB}(x_0)\, J^{AB\,,I}(\mathcal{M}) \tag{6.20}$$

where the various sources are defined in (5.18). This expression represents the non-abelian extension of the action given for example in refs. [18, 19].

We think that our analysis clarifies the role played by D-instantons on the scattering amplitudes of four-dimensional gauge theories already discussed in the literature. In particular we have shown that the stringy procedure to compute instanton corrections to correlation functions reproduces in the field theory limit the standard instanton calculus in virtue of the identification (6.16). We hope that these ideas and techniques can be useful also for practical calculations in the $\mathcal{N} = 4$ SYM theory considered in this paper as well as in gauge theories with lower supersymmetries.

Acknowledgments

We thank Rodolfo Russo for useful discussions and exchange of ideas. This work supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2000-00131 Quantum Spacetime in which A.L.is associated with University of Torino. It is also partially supported by the italian MIUR under the program "Teoria dei Campi, Superstringhe e Gravitá"

[14]For the gauge field A^I_μ there is also an explicit momentum factor, see eq. (4.4).

A. Notations and conventions

Notations: we use the following notations for indices:

- $d = 10$ vector indices: $M, N, \ldots \in \{1, \ldots, 10\}$;
- $d = 4$ vector indices: $\mu, \nu, \ldots \in \{1, \ldots, 4\}$;
- $d = 6$ vector indices: $a, b, \ldots \in \{5, \ldots, 10\}$;
- chiral and anti-chiral spinor indices in $d = 10$: $\mathcal{A}$ and $\dot{\mathcal{A}}$;
- chiral and anti-chiral spinor indices in $d = 4$: α and $\dot{\alpha}$;
- spinor indices in $d = 6$: ^{A} and ${}_{A}$ in the fundamental and anti-fundamental of $\mathrm{SU}(4) \simeq \mathrm{SO}(6)$.

Our choice for the group indices is the following:

- $\mathrm{SU}(N)$ colour indices: $I, J, \ldots \in \{1, \ldots, N^2 - 1\}$;
- $\mathrm{U}(k)$ colour indices: $U, V, \ldots \in \{1, \ldots, k^2\}$;
- D3 indices: $u, v, \ldots \in \{1, \ldots, N\}$;
- D(-1) indices: $i, j, \ldots \in \{1, \ldots, k\}$;
- $\mathrm{SU}(2)$ adjoint indices: $c, d, \ldots \in \{1, 2, 3\}$.

$d = 4$ **Clifford algebra:** the euclidean Lorentz group $\mathrm{SO}(4) \sim \mathrm{SU}(2)_+ \times \mathrm{SU}(2)_-$ is realized on spinors in terms of the matrices $(\sigma^\mu)_{\alpha\dot{\beta}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$ with

$$\sigma^\mu = (\mathbf{1}, -\mathrm{i}\vec{\tau})\,, \qquad \bar{\sigma}^\mu = \sigma_\mu^\dagger = (\mathbf{1}, \mathrm{i}\vec{\tau})\,, \tag{A.1}$$

where τ^c are the ordinary Pauli matrices. They satisfy the Clifford algebra

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = 2\delta_{\mu\nu}\,\mathbf{1}\,, \tag{A.2}$$

and correspond to a Weyl representation of the γ-matrices,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A.3}$$

acting on the spinor

$$\psi = \begin{pmatrix} \psi_\alpha \\ \psi^{\dot{\alpha}} \end{pmatrix}. \tag{A.4}$$

Out of these matrices, the $\mathrm{SO}(4)$ generators are defined by

$$\sigma_{\mu\nu} = \frac{1}{2}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu)\,, \qquad \bar{\sigma}_{\mu\nu} = \frac{1}{2}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu); \tag{A.5}$$

the matrices $\sigma_{\mu\nu}$ are self-dual and thus generate the $\mathrm{SU}(2)_+$ factor; the anti-self-dual matrices $\bar{\sigma}_{\mu\nu}$ generate instead the $\mathrm{SU}(2)_-$ factor. Notice that the indices in the $\mathbf{2}$ of $\mathrm{SU}(2)_+$

JHEP02(2003)045

are denoted by α and those for the **2** of $\mathrm{SU}(2)_-$ by $\dot\alpha$. The charge conjugation matrix is block-diagonal in this Weyl basis:

$$C_{(4)} = \begin{pmatrix} C^{\alpha\beta} & 0 \\ 0 & C_{\dot\alpha\dot\beta} \end{pmatrix} = \begin{pmatrix} -\varepsilon^{\alpha\beta} & 0 \\ 0 & -\varepsilon_{\dot\alpha\dot\beta} \end{pmatrix} \tag{A.6}$$

with $\varepsilon^{12} = \varepsilon_{12} = -\varepsilon^{\dot1\dot2} = -\varepsilon_{\dot1\dot2} = +1$. Moreover we raise and lower spinor indices as follows

$$\psi^\alpha = \varepsilon^{\alpha\beta}\,\psi_\beta\,, \qquad \psi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\,\psi^{\dot\beta}\,. \tag{A.7}$$

't Hooft symbols: the explicit mapping of a self-dual SO(4) tensor into the adjoint representation of the $\mathrm{SU}(2)_+$ factor is realized by the 't Hooft symbols $\eta^c_{\mu\nu}$; the analogous mapping of an anti-self dual tensor into the adjoint of the $\mathrm{SU}(2)_-$ subgroup is realized by $\bar\eta^c_{\mu\nu}$. One has

$$(\sigma_{\mu\nu})_\alpha^{\ \beta} = \mathrm{i}\,\eta^c_{\mu\nu}\,(\tau^c)_\alpha^{\ \beta}\,, \qquad (\bar\sigma_{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \mathrm{i}\,\bar\eta^c_{\mu\nu}\,(\tau^c)^{\dot\alpha}_{\ \dot\beta}\,. \tag{A.8}$$

An explicit representation of the 't Hooft symbols is given by

$$\begin{aligned} \eta^c_{\mu\nu} &= \bar\eta^c_{\mu\nu} = \varepsilon_{c\mu\nu}, \qquad \mu,\nu \in \{1,2,3\}, \\ \bar\eta^c_{4\nu} &= -\eta^c_{4\nu} = \delta^c_\nu, \\ \eta^c_{\mu\nu} &= -\eta^c_{\nu\mu}, \qquad \bar\eta^c_{\mu\nu} = -\bar\eta^c_{\nu\mu}\,. \end{aligned} \tag{A.9}$$

From it one can easily see that

$$\eta^c_{\mu\nu}\,\eta^{d\,\mu\nu} = 4\,\delta^{cd}\,, \tag{A.10}$$

$$\eta^c_{\mu\nu}\,\eta^c_{\rho\sigma} = \delta_{\mu\rho}\,\delta_{\nu\sigma} - \delta_{\mu\sigma}\,\delta_{\nu\rho} + \varepsilon_{\mu\nu\rho\sigma}\,. \tag{A.11}$$

Analogous formulas hold for the contractions of two $\bar\eta$'s with a minus sign in the ε term of (A.11).

$d = 6$ **Clifford algebra:** taking advantage of the equivalence $\mathrm{SO}(6) \sim \mathrm{SU}(4)$, upon which a positive (negative) chirality spinor corresponds to a fundamental (anti-fundamental) SU(4) representation, we can represent the SO(6) spinor as

$$\Lambda = \begin{pmatrix} \Lambda^A \\ \Lambda_A \end{pmatrix} \tag{A.12}$$

on which the following gamma matrices act

$$\Gamma^a = \begin{pmatrix} 0 & \Sigma^a \\ \bar\Sigma^a & 0 \end{pmatrix}\,. \tag{A.13}$$

The matrices Σ^a and $\bar\Sigma^a$ realize the six-dimensional Clifford algebra

$$(\Sigma^a)^{AB}(\bar\Sigma^b)_{BC} + (\Sigma^b)^{AB}(\bar\Sigma^a)_{BC} = 2\,\delta^{ab}\,\delta^A_{\ C}\,, \tag{A.14}$$

(with $(\bar\Sigma^a)_{AB} = (\Sigma^{a\,BA})^*$). An explicit realization can be given in terms of 't Hooft symbols

$$\Sigma^a = \left(\eta^3, \mathrm{i}\bar\eta^3, \eta^2, \mathrm{i}\bar\eta^2, \eta^1, \mathrm{i}\eta^1\right), \qquad \bar\Sigma^a = \left(-\eta^3, \mathrm{i}\bar\eta^3, -\eta^2, \mathrm{i}\bar\eta^2, -\eta^1, \mathrm{i}\eta^1\right). \tag{A.15}$$

The charge conjugation matrix is off-diagonal in this chiral basis:

$$C_{(6)} = \begin{pmatrix} 0 & C_A^{\ B} \\ C^A_{\ B} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}\,\delta_A^{\ B} \\ -\mathrm{i}\,\delta^A_{\ B} & 0 \end{pmatrix}\,. \tag{A.16}$$

$d=10$ **Clifford algebra:** the ten-dimensional γ-matrices $\Gamma^M_{(10)}$ and the charge conjugation matrix $C_{(10)}$ are expressed in terms of the four- and six-dimensional matrices as

$$\begin{aligned} \Gamma^\mu_{(10)} &= \gamma^\mu \otimes \mathbf{1}\,, & \Gamma^a_{(10)} &= \gamma^5 \otimes \Gamma^a\,, \\ \Gamma^{11}_{(10)} &= \gamma^5 \otimes \Gamma^7\,, & C_{(10)} &= C_{(4)} \otimes C_{(6)}\,, \end{aligned} \tag{A.17}$$

such that

$$C_{(10)}\Gamma^M_{(10)}C^{-1}_{(10)} = -\Gamma^{M\,T}_{(10)}\,. \tag{A.18}$$

Spin field correlators: From the general formulae of [34]:

$$\begin{aligned} \mathcal{S}^{(+1/2)}_{\mathcal{A}}(z)\mathcal{S}^{(-1/2)}_{\dot{\mathcal{B}}}(w) &\sim \frac{1}{z-w}C^{\mathcal{A}\dot{\mathcal{B}}} \\ \mathcal{S}^{(+1/2)}_{\mathcal{A}}(z)\mathcal{S}^{(-1/2)}_{\mathcal{B}}(w) &\sim \frac{i}{(z-w)^{1/2}}\psi^{\mathcal{M}}\Gamma^{\mathcal{AB}}_{\mathcal{M}} \end{aligned}$$

by decomposing the ten-dimensional fields into four-dimensional and six-dimensional ones, we can derive the following "effective" OPE's:

$$\begin{aligned} S^{\dot\alpha}(z)\,S_\beta(w) &\sim \frac{1}{\sqrt{2}}\,(\bar\sigma^\mu)^{\dot\alpha}{}_\beta\,\psi_\mu(w)\,, & S^A(z)\,S_B(w) &\sim \frac{\mathrm{i}\,\delta^A_B}{(z-w)^{3/4}}\,, \\ S^{\dot\alpha}(z)\,S^{\dot\beta}(w) &\sim -\frac{\varepsilon^{\dot\alpha\,\dot\beta}}{(z-w)^{1/2}}\,, & S^A(z)\,S^B(w) &\sim \frac{\mathrm{i}}{\sqrt{2}}\,\frac{(\Sigma^a)^{AB}\,\psi_a(w)}{(z-w)^{1/4}}\,, \\ S_\alpha(z)\,S_\beta(w) &\sim \frac{\varepsilon_{\alpha\beta}}{(z-w)^{1/2}}\,, & \psi^a(z)\,S_A(w) &\sim \frac{1}{\sqrt{2}}\,\frac{(\bar\Sigma^a)_{AB}\,S^B(w)}{(z-w)^{1/2}}\,, \\ \psi^\mu(z)\,S^{\dot\alpha}(w) &\sim \frac{1}{\sqrt{2}}\,\frac{(\bar\sigma^\mu)^{\dot\alpha\beta}\,S_\beta(w)}{(z-w)^{1/2}}\,, & \psi^a(z)\,S^A(w) &\sim -\frac{1}{\sqrt{2}}\,\frac{(\Sigma^a)^{AB}\,S_B(w)}{(z-w)^{1/2}}\,, \\ \psi^\mu\psi^\nu(z)\,S^{\dot\alpha}(w) &\sim -\frac{1}{2}\,\frac{(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,S^{\dot\beta}(w)}{(z-w)}\,, & \psi^a\psi^b(z)\,S^A(w) &\sim \frac{1}{2}\,\frac{(\Sigma^{ab})^A{}_B\,S^B(w)}{(z-w)}\,. \end{aligned} \tag{A.19}$$

Other OPE's which do not appear in (A.19) can be simply obtained by a suitable change of the chiralities. From these OPE's we can derive the following 3-point functions which have been used in the main text

$$\begin{aligned} \langle S^{\dot\alpha}(z_1)\psi_\mu(z_2)S_\beta(z_3)\rangle &= \frac{1}{\sqrt{2}}\,(\bar\sigma_\mu)^{\dot\alpha}{}_\beta(z_1-z_2)^{-1/2}(z_2-z_3)^{-1/2}\,, \\ \langle S^{\dot\alpha}(z_1)\psi_\mu\psi_\nu(z_2)S^{\dot\beta}(z_3)\rangle &= -\frac{1}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}(z_1-z_3)^{1/2}(z_1-z_2)^{-1}(z_2-z_3)^{-1}\,, \\ \langle S^A(z_1)\psi^a(z_2)\,S^B(z_3)\rangle &= \frac{\mathrm{i}}{\sqrt{2}}\,(\Sigma^a)^{AB}(z_1-z_2)^{-1/2}(z_1-z_3)^{-1/4}(z_2-z_3)^{-1/2}\,, \\ \langle S_A(z_1)\psi^a(z_2)S_B(z_3)\rangle &= -\frac{\mathrm{i}}{\sqrt{2}}\,(\bar\Sigma^a)_{AB}(z_1-z_2)^{-1/2}(z_1-z_3)^{-1/4}(z_2-z_3)^{-1/2}\,. \end{aligned} \tag{A.20}$$

Twist field correlators: the $(-1)/3$ and the $3/(-1)$ strings have four Neumann-Dirichlet directions, namely those along the world-volume of the D3 branes. Thus, the string fields X^μ have twisted boundary conditions; this fact can be seen as due to the presence of twist and anti-twist fields $\Delta(z)$ and $\bar{\Delta}(z)$ that change the boundary conditions from Neumann to Dirichlet and vice-versa by introducing a cut in the world-sheet (see for example ref. [27]). The twist fields $\Delta(z)$ and $\bar{\Delta}(z)$ are bosonic operators with conformal dimension 1/4 and their OPE's are

$$\Delta(z_1)\,\bar{\Delta}(z_2) \sim (z_1 - z_2)^{1/2}\,, \qquad \bar{\Delta}(z_1)\,\Delta(z_2) \sim -\,(z_1 - z_2)^{1/2}\,, \tag{A.21}$$

where the minus sign in the second correlator is again an "effective" rule to correctly account for the space-time statistics in correlation functions.

B. A short review of the ADHM construction and of zero modes around an instanton background

Following the notation of refs. [22, 20], we begin by introducing the basic objects in the ADHM construction of the SU(N) instanton solution in four dimensions, namely the $[N+2k]\times[2k]$ and $[2k]\times[N+2k]$ matrices

$$\Delta(x) = a + bx\,, \qquad \bar{\Delta}(x) = \bar{a} + \bar{x}\bar{b} \tag{B.1}$$

where $x_{\alpha\dot{\beta}} = x_\mu\,(\sigma^\mu)_{\alpha\dot{\beta}}$ and $\bar{x}^{\dot{\alpha}\beta} = x_\mu\,(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$ describe the position of the multi-instanton center of mass, and all the remaining moduli are collected in the matrix a (see formula (B.3) below). Finally, b is a $[N+2k]\times[2k]$ matrix which can be conveniently chosen to be

$$b = \begin{pmatrix} 0 \\ \mathbf{1}_{[2k]\times[2k]} \end{pmatrix}, \qquad \bar{b} = \left(0, \mathbf{1}_{[2k]\times[2k]}\right). \tag{B.2}$$

The moduli space of the solutions to the self-dual equations of motion is characterized in terms of the supercoordinates

$$a \equiv \begin{pmatrix} w_{\dot{\alpha}}^{\ ui} \\ a'_{\alpha\dot{\beta}\,li} \end{pmatrix}, \qquad \mathcal{M}^A \equiv \begin{pmatrix} \mu^{Aui} \\ M'^{\beta A}_{\ \ li} \end{pmatrix}, \tag{B.3}$$

which satisfy the bosonic and fermionic ADHM constraints

$$\bar{\Delta}\,\Delta = f^{-1}_{k\times k}\,\mathbf{1}_{[2]\times[2]}\,, \tag{B.4}$$

$$\bar{\Delta}\mathcal{M}^A = \bar{\mathcal{M}}^A\,\Delta \tag{B.5}$$

with $f_{k\times k}$ an invertible $k\times k$ matrix.

The solutions to the self-dual equations of motion for the various fields in the $\mathcal{N}=4$ vector multiplet are given by

$$\begin{aligned} \widehat{A}_\mu &= \bar{U}\,\partial_\mu U\,, \\ \widehat{\Lambda}^A &= \bar{U}\left(\mathcal{M}^A f\,\bar{b} - b\,f\,\bar{\mathcal{M}}^A\right)U\,, \\ \widehat{\varphi}^{AB} &= -\frac{\mathrm{i}}{2\sqrt{2}}\,\bar{U}\left(\mathcal{M}^B f\,\bar{\mathcal{M}}^A - \mathcal{M}^A f\,\bar{\mathcal{M}}^B\right)U - \\ &\quad -\mathrm{i}\,\bar{U}\cdot\begin{pmatrix} 0_{[N]\times[N]} & 0_{[N]\times[2k]} \\ 0_{[2k]\times[N]} & L^{-1}\Lambda^{AB}_{[k]\times[k]}\otimes 1_{[2]\times[2]} \end{pmatrix}\cdot U\,, \end{aligned} \tag{B.6}$$

in terms of the kernels $U_{[N+2k]\times[N]}$ and $\bar{U}_{[N]\times[N+2k]}$ of the ADHM matrices $\bar{\Delta}$ and Δ. In (B.6), the hatted gauge fields are taken to be anti-hermitian, Λ^{AB} is the fermionic bilinear

$$\Lambda^{AB} = \frac{1}{2\sqrt{2}} \left(\bar{\mathcal{M}}^A \mathcal{M}^B - \bar{\mathcal{M}}^B \mathcal{M}^A \right) , \tag{B.7}$$

and the operator L is defined as

$$L \cdot \Omega = \frac{1}{2}\{W^0, \Omega\} + [a_\mu, [a^\mu, \Omega]] , \tag{B.8}$$

with $(W^0)_j^{\ i} = w_{\dot{\alpha}}^{\ ui}\, \bar{w}^{\dot{\alpha}}_{\ uj}$.

For simplicity, from now on we concentrate on solutions with winding number $k = 1$, which for SU(N) can be found starting from those for SU(2). For $k = 1$ the ADHM constraints drastically simplify; indeed, the bosonic constraint (B.4) simply reduces to

$$\bar{w}^{\dot{\alpha}}_{\ u}\, w_{\dot{\beta}}^{\ u} = \rho^2\, \delta^{\dot{\alpha}}_{\dot{\beta}} \tag{B.9}$$

(see eq. (4.8)), which is solved by

$$\| w_{\dot{\alpha}}^{\ u} \| = \left\| \bar{w}^{\dot{\alpha}}_{\ u} \right\| = \rho\, T \begin{pmatrix} 0_{[N-2]\times[2]} \\ 1_{[2]\times[2]} \end{pmatrix} \tag{B.10}$$

where $T \in \mathrm{SU}(N)/\,\mathrm{SU}(N-2)$. This is just the standard SU(2) instanton solution embedded inside the SU(N) in the lower right corner. The matrices T describe the orientation of the SU(2) instanton inside SU(N) with SU($N-2$) being the stability group of the SU(2) instanton solution. If we temporarily set $T = 1$, the vector field, which solves the equations of motion in the singular gauge, can be written as

$$(\widehat{A}_\mu)^u_{\ v} = \frac{\rho^2}{x^2\,(x^2+\rho^2)}\, (\bar{\sigma}_{\nu\mu})^u_{\ v}\, x^\nu , \tag{B.11}$$

where

$$(\bar{\sigma}_{\nu\mu})_u^{\ v} = \begin{pmatrix} 0_{[N-2]\times[N-2]} & 0_{[N-2]\times[2]} \\ 0_{[2]\times[N-2]} & (\bar{\sigma}_{\nu\mu})_{\dot{\alpha}}^{\ \dot{\beta}} \end{pmatrix} , \tag{B.12}$$

and the center of the instanton has been set at $x_0 = 0$ for simplicity. If we remove the $T = 1$ constraint and shift the instanton center, we find the general SU(N) solution $\widehat{A}_\mu = T\,\widehat{A}_\mu\, T^{-1}$ which is given in (4.14). As we have also found in the main text, an explicit representation of our embedding is given by the matrices in (4.9) where the $w_{\dot{\alpha}}^{\ u}$'s are chosen according to (B.10).

We now turn to the fermionic the zero modes. Their number is $2kN\mathcal{N}$ and obviously depends on the number of supersymmetries. For compatibility with the rest of the paper we will discuss the $\mathcal{N} = 4$ case. The $\mathcal{N} = 2$ and $\mathcal{N} = 1$ cases can easily be deduced from our discussion by restricting the range of the capital latin indices in the following to $A, B = 1, 2$ and $A, B = 1$ respectively. It is well-known that in the SU(2) case the fermionic zero modes are in the adjoint representation and that their explicit form can be found by acting with the supersymmetry charges of the superconformal algebra on the instanton solution, leading to

$$\Lambda^{\alpha A} = \frac{\mathrm{i}}{2} \left(\eta^{\beta A} - \bar{\zeta}_{\dot{\gamma}}^{\ A}\, (\bar{\sigma}_\rho)^{\dot{\gamma}\beta}\, x^\rho \right) (\sigma^{\mu\nu})_\beta^{\ \alpha}\, F_{\mu\nu} . \tag{B.13}$$

These solutions can be singled out also for arbitrary winding numbers k, since they correspond to solutions of the constraint (B.5) in which the fermionic matrix $\mathcal{M}^A$ is taken to be proportional to the matrices a, b introduced in (B.1), namely

$$\mu^{Aui} = 0\,, \qquad M'^{\beta A}_{\;\;ij} = b_{ij}\,\eta^{\beta A}\,, \tag{B.14}$$

and

$$\mu^{Aui} = w_{\dot\alpha}^{\;\;ui}\,\bar\zeta^{\dot\alpha A}\,, \qquad M'^{\beta A}_{\;\;ij} = -\bar\zeta_{\dot\alpha}^{\;\;A}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,a'_{\;\mu ij}\,, \tag{B.15}$$

for the supersymmetric and superconformal zero-modes respectively.

Besides the zero-modes (B.13), in the SU(N) case we have other $4\mathcal{N}(N-2)$ fermionic zero-modes, which are the partners of the color rotations parametrized by $w_{\dot\alpha}^{\;\;u}$'s. They transform in the fundamental representation of the embedded SU(2) and correspond to the $2(N-2)$ doublets in the decomposition of the adjoint representation of SU(N) with respect to SU(2). For example, for SU(3) we have $\mathbf{8} = \mathbf{3} \oplus \mathbf{2} \oplus \bar{\mathbf{2}} \oplus \mathbf{1}$. Since there are no solutions to the Dirac equation which are SU(2) singlets, and since we already know the form (B.13) of the solution in the adjoint representation, we simply have to recall the form of the SU(2) solutions in the fundamental. They are

$$\psi_{\alpha s} = \frac{\rho\,\epsilon_{\alpha s}}{\sqrt{(x^2+\rho^2)^3}} \tag{B.16}$$

where $s = 1, 2$ is an index which runs in the fundamental. The solutions for $\bar{\mathbf{2}}$ are obtained from those in (B.16) by raising the indices α and s. Let us now turn to the SU(N) case and introduce the gauge invariant quantity $(W^{\dot\alpha}_{\dot\beta})^j_{\;i} = \bar w^{\dot\alpha}_{\;\;ui}\,w_{\dot\beta}^{\;\;uj}$. By definition, the infinitesimal gauge rotations which leave this quantity invariant are those which satisfy

$$\delta\bar w^{\dot\alpha}_{\;\;ui}\,w_{\dot\beta}^{\;\;uj} + \bar w^{\dot\alpha}_{\;\;ui}\,\delta w_{\dot\beta}^{\;\;uj} = 0\,. \tag{B.17}$$

Using for δw and $\delta\bar w$ the transformations (2.35), from (B.17) we get

$$\bar\xi^{\dot\alpha}_{\;\;A}\,\bar\mu^A_{\;\;ui}\,w_{\dot\beta}^{\;\;uj} + \bar\xi_{\dot\beta A}\,\bar w^{\dot\alpha}_{\;\;ui}\,\mu^{Auj} = 0\,, \tag{B.18}$$

from which we infer

$$\bar\mu^A_{\;\;ui}\,w_{\dot\beta}^{\;\;uj} = 0\,, \qquad \bar w^{\dot\alpha}_{\;\;ui}\mu^{Auj} = 0\,. \tag{B.19}$$

For $k=1$, given the choice eq. (B.9), this implies $\mu^{Au} = (\mu_1^A, \ldots, \mu_{N-2}^A, 0, 0)$. Starting from (B.16) we can now deduce the SU(N) formulae by replacing the index s in the fundamental of SU(2) with an index v in the fundamental of SU(N), and adding another index u to label the $N-2$ different solutions. For convenience the range of u will be extended to N. For consistency with our previous notation, we also substitute ϵ with μ. Putting together doublets and anti-doublets, we finally find

$$(\widehat\Lambda^{\dot\alpha A})^u_{\;\;v}\Big|_{\rm reg.} = \frac{\rho}{\sqrt{(x^2+\rho^2)^3}}\,(\mu^{Au}\,\delta^{\dot\alpha}_{\;\;v} + \varepsilon^{\dot\alpha u}\,\bar\mu^A_{\;\;v}) \tag{B.20}$$

where $\varepsilon^{\dot\alpha u} = (0, \ldots, 0, \varepsilon^{\dot\alpha\dot\beta})$ is a natural extension of the Levi-Civita symbol to our case. To go to the singular gauge we perform a SU(N) gauge transformation extending the standard

JHEP02(2003)045

SU(2) one, i.e. $g = x_\mu \sigma^\mu / \sqrt{x^2}$, to $g' = (0, \ldots, 0, x_\mu \sigma^\mu)/\sqrt{x^2}$, and get

$$(\widehat{\Lambda}^{\alpha A})^u{}_v = \frac{\rho}{\sqrt{x^2(x^2+\rho^2)^3}} \, (\mu^{Au} \, x^\alpha{}_v + x^{\alpha u} \bar{\mu}^A{}_v) \,, \tag{B.21}$$

where $x^\alpha{}_v = (0, \ldots, 0, x_\mu \, (\sigma^\mu)^\alpha{}_{\dot\beta})$.

At last we discuss the inhomogeneous solutions of the equations of motion for the adjoint scalars $\widehat{\varphi}^{AB}$. These equations follow from the SYM action (3.11)[15] and are

$$\mathcal{D}^2 \, \widehat{\varphi}^{AB} - \frac{1}{\sqrt{2}} \left\{ \widehat{\Lambda}^{\alpha A} \, , \, \widehat{\Lambda}_\alpha{}^B \right\} + \cdots = 0 \,, \tag{B.22}$$

where the ellipses stand for terms that contain $\bar{\Lambda}^{\alpha A}$ or are trilinear in the scalar fields, which are not relevant for our present analysis. A first part of the solution of (B.22) is obtained by using for $\widehat{\Lambda}^{\alpha A}$ the supersymmetric zero-modes (B.13). This leads to

$$(\widehat{\varphi}^{AB})^u{}_v = \frac{4\sqrt{2}}{(x^2+\rho^2)^2} \eta^{[Au} \eta^{B]}_{\ \ v} \,. \tag{B.23}$$

In the SU(N) case there is an additional contribution to (B.23) coming from the zero modes (B.21). For $k = 1$ and $T = 1$, it is easy to see that

$$\left\{ \widehat{\Lambda}^{\alpha A}, \widehat{\Lambda}_\alpha{}^B \right\}^u{}_v = \frac{4\rho^2}{(x^2+\rho^2)^3} \left(\mu^{[Au} \bar{\mu}^{B]}_{\ \ v} - \frac{1}{2} \mu^{[Ap} \, \bar{\mu}^{B]}_{\ \ p} \, \tilde{\delta}^u{}_v \right) \tag{B.24}$$

where $\tilde{\delta}^u{}_v$ is defined in (5.17). Substituting the tentative solution

$$(\widehat{\varphi}^{AB})^u{}_v = f(x, \rho) \left(\mu^{[Au} \bar{\mu}^{B]}_{\ \ v} - \frac{1}{2} \, \mu^{[Ap} \, \bar{\mu}^{B]}_{\ \ p} \, \tilde{\delta}^u{}_v \right) \tag{B.25}$$

in (B.22) and solving the resulting differential equation for $f(x, \rho)$, one obtains

$$f(x, \rho) = - \frac{1}{2\sqrt{2}(x^2+\rho^2)} \,. \tag{B.26}$$

C. Subleading order of the instanton profile in the $\alpha' \to 0$ limit

In section 4.1 we mentioned that the subleading terms in the large distance expansion of the instanton solution are naturally associated to mixed disks with more insertions of boundary changing operators (see figure 3), and that in the limit $\alpha' \to 0$ they reduce to simple tree-level field theory diagrams, in complete analogy with the gravitational brane solutions as discussed in ref. [32]. As an example, in this appendix we explicitly compute the second order contribution to the gauge field, which is represented by the diagram in figure 4. For simplicity we just consider the SU(2) case. The necessary ingredients to compute this diagram are:

[15]We recall that the fields appearing in the SYM action (3.11) are hermitian, while the hatted fields we are now considering are anti-hermitian; the precise relation between the two is given by $\widehat{\varphi}^{AB} = -\mathrm{i} \, \varphi^{AB}$ and similarly for the other components of the supermultiplet.

- the ordinary 3-gluon vertex of YM theory

$$V^{cde}_{\mu\nu\lambda}(p,q,k) = \mathrm{i}\,\varepsilon^{cde}\Big[(q-k)_\mu\,\delta_{\nu\lambda} + (p-q)_\lambda\,\delta_{\mu\nu} + (k-p)_\nu\,\delta_{\lambda\mu}\Big] \tag{C.1}$$

  where all momenta are incoming, and

- the source subdiagram representing the leading order expression of the gauge field in momentum space given in (4.4), namely

$$A^c_\mu(p;\rho)^{(1)} = \mathrm{i}\rho^2\,\bar{\eta}^c_{\nu\mu}\,p^\nu\,\mathrm{e}^{-\mathrm{i}p\cdot x_0}\,. \tag{C.2}$$

The amplitude in figure 4 is then obtained by sewing two first-order diagrams to a 3-gluon vertex and reversing the sign of the momentum of the free gluon line to describe an *outgoing* field. Taking into account a simmetry factor of 1/2, we have

$$\begin{aligned} A^c_\mu(p;\rho)^{(2)} &= \frac{1}{2}\int\frac{d^4q}{(2\pi)^2}\left[V^{cde}_{\mu\nu\lambda}(-p,q,p-q)\frac{1}{q^2}\,A^d_\nu(q;\rho)^{(1)}\frac{1}{(p-q)^2}A^e_\lambda(p-q;\rho)^{(1)}\right] \\ &= \frac{\mathrm{i}}{2}\rho^4\epsilon^{cde}\bar{\eta}^d_{\sigma\nu}\bar{\eta}^e_{\tau\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x_0}\int\frac{d^4q}{(2\pi)^2}\frac{1}{q^2(p-q)^2}q^\sigma\,(p-q)^\tau\,\times \\ &\qquad\times\Big[(p-2q)_\mu\delta_{\nu\lambda} + (q+p)_\lambda\delta_{\mu\nu} + (q-2p)_\nu\delta_{\lambda\mu}\Big] \end{aligned} \tag{C.3}$$

where the momentum integral can be computed in dimensional regularization. To obtain the space-time profile, we take the Fourier transform of $A^c_\mu(p;\rho)^{(2)}$ multiplied by $1/p^2$, and after some standard manipulations we find

$$(A^c_\mu(x))^{(2)} \equiv \lim_{d\to 4}\int\frac{d^dp}{(2\pi)^{d/2}}\,(A^c_\mu(p;\rho))^{(2)}\,\frac{1}{p^2}\,\mathrm{e}^{\mathrm{i}p\cdot x} = -2\rho^4\bar{\eta}^c_{\mu\nu}\frac{(x-x_0)^\nu}{(x-x_0)^6}\,, \tag{C.4}$$

which is exactly the second order term in eq. (4.12). The higher order terms in the large distance expansion can in principle be computed in a similar manner and thus the full instanton solution can eventually be reconstructed.

References

[1] Z. Bern, L.J. Dixon and D.A. Kosower, *Progress in one-loop QCD computations*, *Ann. Rev. Nucl. Part. Sci.* **46** (1996) 109 [hep-ph/9602280];
Z. Bern, *Perturbative quantum gravity and its relation to gauge theory*, *Living Rev. Rel.* **5** (2002) 5 [gr-qc/0206071], and references therein.

[2] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [hep-th/9510017].

[3] J. Polchinski, *Lectures on D-branes*, hep-th/9611050;
C.V. Johnson, *D-brane primer*, hep-th/0007170.

[4] C.G. Callan Jr., C. Lovelace, C.R. Nappi and S.A. Yost, *Loop corrections to superstring equations of motion*, *Nucl. Phys.* **B 308** (1988) 221;
J. Polchinski and Y. Cai, *Consistency of open superstring theories*, *Nucl. Phys.* **B 296** (1988) 91.

JHEP02(2003)045

[5] P. Di Vecchia and A. Liccardo, *D-branes in string theory, I*, hep-th/9912161; *D-branes in string theory, II*, hep-th/9912275.

[6] P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda and R. Russo, *Classical p-branes from boundary state*, *Nucl. Phys.* **B 507** (1997) 259 [hep-th/9707068].

[7] P. Di Vecchia, M. Frau, A. Lerda and A. Liccardo, (F, Dp) *bound states from the boundary state*, *Nucl. Phys.* **B 565** (2000) 397 [hep-th/9906214].

[8] E. Witten, *Bound states of strings and p-branes*, *Nucl. Phys.* **B 460** (1996) 335 [hep-th/9510135].

[9] M.R. Douglas, *Gauge fields and D-branes*, *J. Geom. Phys.* **28** (1998) 255 [hep-th/9604198]; *Branes within branes*, hep-th/9512077.

[10] J. Polchinski, *Combinatorics of boundaries in string theory*, *Phys. Rev.* **D 50** (1994) 6041 [hep-th/9407031].

[11] M.B. Green and M. Gutperle, *Effects of d-instantons*, *Nucl. Phys.* **B 498** (1997) 195 [hep-th/9701093]; *D-particle bound states and the d-instanton measure*, *J. High Energy Phys.* **01** (1998) 005 [hep-th/9711107]; *D-instanton partition functions*, *Phys. Rev.* **D 58** (1998) 046007 [hep-th/9804123].

[12] T. Banks and M.B. Green, *Non-perturbative effects in* $AdS_5 \times S^5$ *string theory and* $D = 4$ *SUSY Yang-Mills*, *J. High Energy Phys.* **05** (1998) 002 [hep-th/9804170].

[13] C.-S. Chu, P.-M. Ho and Y.-Y. Wu, *D-instanton in* AdS_5 *and instanton in* SYM_4, *Nucl. Phys.* **B 541** (1999) 179 [hep-th/9806103].

[14] I.I. Kogan and G. Luzon, *D-instantons on the boundary*, *Nucl. Phys.* **B 539** (1999) 121 [hep-th/9806197].

[15] M. Bianchi, M.B. Green, S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and d-instantons in IIB superstring theory*, *J. High Energy Phys.* **08** (1998) 013 [hep-th/9807033].

[16] N. Dorey, V.V. Khoze, M.P. Mattis and S. Vandoren, *Yang-Mills instantons in the large-N limit and the AdS/CFT correspondence*, *Phys. Lett.* **B 442** (1998) 145 [hep-th/9808157]; *Yang-Mills instantons in the large-N limit and the AdS/CFT correspondence*, *Phys. Lett.* **B 442** (1998) 145 [hep-th/9808157].

[17] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in* $N = 4$ *superconformal field theory*, *Nucl. Phys.* **B 552** (1999) 88 [hep-th/9901128].

[18] M.B. Green and M. Gutperle, *D-instanton induced interactions on a D3-brane*, *J. High Energy Phys.* **02** (2000) 014 [hep-th/0002011].

[19] N. Dorey, T.J. Hollowood and V.V. Khoze, *Notes on soliton bound-state problems in gauge theory and string theory*, hep-th/0105090.

[20] A.V. Belitsky, S. Vandoren and P. van Nieuwenhuizen, *Yang-Mills and d-instantons*, *Class. and Quant. Grav.* **17** (2000) 3521 [hep-th/0004186].

[21] N. Dorey, T.J. Hollowood and V.V. Khoze, *A brief history of the stringy instanton*, hep-th/0010015.

[22] N. Dorey, T.J. Hollowood, V.V. Khoze and M.P. Mattis, *The calculus of many instantons*, *Phys. Rept.* **371** (2002) 231 [hep-th/0206063].

[23] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[24] A.A. Belavin, A.M. Polyakov, A.S. Shvarts and Y.S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, *Phys. Lett.* **B 59** (1975) 85.

[25] G. 't Hooft, *Computation of the quantum effects due to a four-dimensional pseudoparticle*, *Phys. Rev.* **D 14** (1976) 3432, erratum *ibid.* **D 18** (1978) 2199.

[26] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[27] L.J. Dixon, J.A. Harvey, C. Vafa and E. Witten, *Strings on orbifolds*, *Nucl. Phys.* **B 261** (1985) 678;
S. Hamidi and C. Vafa, *Interactions on orbifolds*, *Nucl. Phys.* **B 279** (1987) 465;
L.J. Dixon, D. Friedan, E.J. Martinec and S.H. Shenker, *The conformal field theory of orbifolds*, *Nucl. Phys.* **B 282** (1987) 13.

[28] P. Di Vecchia, L. Magnea, A. Lerda, R. Russo and R. Marotta, *String techniques for the calculation of renormalization constants in field theory*, *Nucl. Phys.* **B 469** (1996) 235 [hep-th/9601143].

[29] D. Polyakov, *Brst properties of new superstring states*, *Phys. Rev.* **D 65** (2002) 084041 [hep-th/0111227].

[30] A. Sen, *Type I d-particle and its interactions*, *J. High Energy Phys.* **10** (1998) 021 [hep-th/9809111].

[31] L. Gallot, A. Lerda and P. Strigazzi, *Gauge and gravitational interactions of non-BPS d-particles*, *Nucl. Phys.* **B 586** (2000) 206 [hep-th/0001049].

[32] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda, R. Marotta and R. Russo, *Is a classical description of stable non-BPS D-branes possible?*, *Nucl. Phys.* **B 590** (2000) 471 [hep-th/0007097].

[33] M. Billó, P. Di Vecchia, M. Frau, A. Lerda, I. Pesando, R. Russo and S. Sciuto, *Microscopic string analysis of the D0-D8 brane system and dual RR states*, *Nucl. Phys.* **B 526** (1998) 199 [hep-th/9802088].

[34] V.A. Kostelecky, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, *Conformal techniques, bosonization and tree level string amplitudes*, *Nucl. Phys.* **B 288** (1987) 173.

JHEP02(2003)045